Construction of Spherical Codes Using the Hopf Fibration

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Abstract

A new approach to construct spherical codes in $\mathbb{R}^4$ is presented, based on properties of the Hopf fibration and inspired by a previous construction on layers of flat tori [1][2]. We use the Hopf foliation of the 3-sphere by tori to construct a two-step algorithm: (i) to choose a torus parametrised by height and (ii) to distribute points in each torus by iterated rotation matrices. Our performance matches the previous method and is expected to surpass it in higher dimensions.

Key words:

Spherical codes, Hopf fibration, division algebras.

Introduction

A spherical code $C(M,n)$ is a set of points on the surface of the unit $(n-1)$-dimensional sphere:

$$C(M,n) \coloneqq \{x_1, \dots, x_M\} \subset S^{n-1}.$$

We address the problem of *spherical packing*: given a minimum distance d, to find the largest possible number of points M on S^{n-1} such that the Euclidean distance between any two of them is at least d.

We use the properties of the Hopf fibration h, defined as

$$h\colon S^{2n-1} \to S^n$$
$$(z_0, z_1) \longmapsto (2z_0\overline{z_1}, |z_0|^2 - |z_1|^2)$$

where z_0, z_1 are elements of one of the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (i.e. $n = 1, 2, 4, 8$). Each Hopf map induces a fibre bundle structure $S^{n-1} \hookrightarrow S^{2n-1} \to S^n$.

Results and Discussion

We treated the problem in dimension 4 ($n = 2$), inspired by the construction of Torezzan et al. [1][2] on layers of flat tori. We exploit the fact that the sphere S^3 is foliated by tori T^2 with a natural parametrization given by the Hopf fibration:

$$(\eta, \xi_1, \xi_2) \longmapsto (e^{i\xi_1}\sin\eta, e^{i\xi_2}\cos\eta)$$

where $\eta \in \left[0, \frac{\pi}{2}\right]$ and $\xi_i \in [0, 2\pi[,\ i = 1,2.$

Step 1: Varying η, we find tori T_η mutually distant of at least d, using the fact that the minimum distance between the tori T_{η_1} and T_{η_2} in S^3 coincides with the distance between two points $e^{i\eta_1}$ and $e^{i\eta_2}$ on the first quadrant of S^1.

Step 2: For each torus, we choose n internal circles with m equidistantly distributed points, such that subsequent circles have a phase shift of $\psi_m = \pi/m$.

This method approaches a mapping of points of a hexagonal lattice onto the surface of a torus.

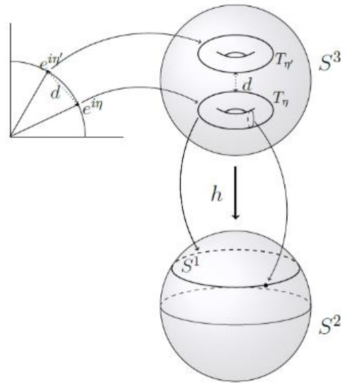

Image 1. Hopf map and distance between tori.

Chart 1. 4-dimensional code sizes at various minimum distances d (* unknown values) [1].

d	Hopf (4D)	TLSC (4,d)	Apple-peeling	Wrapped	Laminated
0.5	152	172	136	*	*
0.4	280	308	268	*	*
0.3	728	798	676	*	*
0.2	2,656	2,718	2,348	*	*
0.1	22,016	22,406	19,364	17,198	16,976
0.01	2.27×10^7	2.27×10^7	1.97×10^7	2.31×10^7	2.31×10^7

Conclusions

Our procedure performs similarly to the method of layers of flat tori (TLSC) by Torezzan et al. [1][2] and it outperforms other methods for some ranges of d. Indeed, in dimension 4, the two constructions are equivalent and the Hopf fibration can be seen as an alternative approach. However, extending the Hopf fibration to dimensions 8 and 16, one expects to outperform known methods in terms of a reasonable compromise between point packing and decoding complexity; this constitutes work in progress.

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