An example of classical variety in projective geometry.

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Abstract
We show that given three pairwise skew lines L, M and N, in the projective space \( \mathbb{P}^3 \), there is only one Segre variety that contains these lines.

Key words:
Algebraic Geometry, Algebra, Projective Geometry

Introduction

The Segre variety is defined as the image of the map \( \sigma \), given by

\[
\sigma: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}
\]

\[
\sigma((X_0, ..., X_n), (Y_0, ..., Y_m)) = (..., X_iY_j, ...),
\]

where the coordinates on the image range over all the pairs of coordinates \( X_i \) and \( Y_j \). We denote the Segre variety by \( \Sigma_{1,1} \).

Our goal is to find a variety that contains the three lines, and is also isomorphic to the Segre variety \( \Sigma_{1,1} \).

Results and Discussion

We define the variety \( X \) as the union of all lines meeting \( L, M \) and \( N \). We will show that there is a regular map \( \varphi: X \to \Sigma_{1,1} \), such that the inverse map of \( \varphi \) is also regular.

We can suppose, by a change of coordinates, that \( L = (e_1, e_2), M = (e_2, e_3) \) and \( N = (v_1, v_2) \). There exist \( 2 \times 2 \) matrices \( M_1 \) and \( M_2 \), such that the block matrix

\[
A = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}
\]

preserves \( L \) and \( M \), and transforms \( N \) in \( (e_1 + e_3, e_2 + e_4) \).

We now consider the map \( \phi \), defined by

\[
\phi((\lambda: \mu), (x_1: x_2)) = \lambda(x_0: x_1: x_0: x_1) + \mu(x_0: x_1: 0: 0).
\]

We observe that for each fixed point \( z_0 = (x_0: y_0) \in \mathbb{P}^1 \), the map \( \phi(z_0) \) is a line in \( \mathbb{P}^3 \) that meet \( L, M \) and \( N \), moreover, all the lines that meets \( L, M \) and \( N \) can be described in this way, then \( \text{Im}(\phi) = X \).

Lastly, we define the regular map \( \varphi \) as

\[
\varphi: X \to \Sigma_{1,1}
\]

\[
\varphi(\lambda(x_0: x_1: x_0: x_1) + \mu(x_0: x_1: 0: 0)) = \lambda(0: 0: x_1: x_1) + \mu(x_0: x_1: 0: 0)
\]

Figure 1. \( \Sigma_{1,1} \) and the lines \( L, M \) and \( N \).

Conclusions

Once it is shown that the inverse of the map \( \varphi \) is regular, we get that, indeed, \( X \) is a variety, and because the map \( \varphi \) is surjective, we conclude that \( X \cong \Sigma_{1,1} \).

The uniqueness is obtained showing that the three lines impose conditions over the coefficients from the quadratic polynomials associated to the Segre variety, in order to find one and only one variety.

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