About the existence of conformal metrics on a closed Riemannian manifold \((M,g)\) in a such way that a closed \(p\)-form \(\omega\) is harmonic.

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Abstract

We present a sufficient condition for the existence of a conformal metric \(h\) on a closed Riemannian manifold \((M,g)\) with a closed \(p\)-form \(\omega\) in a such way that \(\Delta_h \omega = 0\).

Key words: Hodge theory, Laplace operator, harmonic forms.

Introduction

By Hodge theory (1930), for a closed \(p\)-form \(\omega\) be harmonic in a closed Riemannian manifold \((M,g)\) is necessary and sufficient that \(\omega\) be co-closed.

\[ \Delta_g \omega = 0 \iff d \omega = \delta \omega = 0. \]

We know, however, that \(\delta = \ast \cdot \delta \cdot \ast \cdot g\), and the condition "be co-closed" depends on the metric.

This allows ask:

Given a closed \(p\)-form \(\omega\) on a closed Riemannian manifold \((M,g)\), when does there exist a metric \(h\) on \(M\) in a such way that \(\Delta_h \omega = 0\)?

This question was answered completely for 1-forms by Calabi in 1969.

We present a sufficient condition for the solution in the case of \(p\)-forms in each open of some atlas over \(M\).

Results and Discussion

We search for a metric

\[ h = e^{2f} \cdot g \]

where \(f : M \rightarrow \mathbb{R}\) is a \(C^\infty\)function.

Let \(\ast \cdot g\), the Hodge star on the metric \(g\). For \(\omega\) be co-closed on \(h\)-metric:

\[ d \ast_h \omega = 0. \]

Then the condition to \(\omega\) be co-closed is the existence of a function \(f \in C^\infty(M)\) such that

\[ df \wedge \ast_g \omega = -d \ast_g \omega \quad (1) \]

We rewrite it in terms of a matrix equation:

Using the multi-index notation, any \(p\)-form \(\omega\) can be written as:

\[ \omega = \sum_{I} \omega_I dx^I, \]

\[ I = \{i_1 < \cdots < i_p\}. \]

Let \(J\) be the multi-index complementary of \(I\).

Then we define the vector \(S^J\) of \(\mathbb{R}^n\) whose coordinates \(j = 1, \ldots, n\) are:

\[ (S^J)_j = \begin{cases} (-1)^\alpha \omega_{(j-i_k)}, & j = i_k \\ 0, & j \text{ not in } J. \end{cases} \]

where \(\alpha\) means the signal of permutation to rearrange \(J\) as an ascending multi-index.

Let \(A\) the matrix whose entries are \(S^J\), and choose a column vector \(c\) whose components are \(-\nabla . S^J\).

Then we rewrite the PDE (1) as:

\[ \nabla f^t \cdot A = c \quad (2) \]

Conclusions

Once the problem reduces to linear algebra, the sufficient condition for the existence of conformal metric is given by the invertibility of \(A\).

But since \(A\) in invertible we get:

\[ \nabla f^t = c.A^{-1} \]

Then the solution \(f\) is obtained by integration in each open set of an atlas over \(M\).

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