About the existence of conformal metrics on a closed Riemannian manifold (M, g) in a such way that a closed p-form ω is harmonic.

Henrique Nogueira de Sá Earp (PQ), Leonardo Francisco Cavenaghi (IC).

Abstract

We present a sufficient condition for the existence of a conformal metric h on a closed Riemannian manifold (M, g) with a closed p-form ω in a such way that $\Delta_h \omega = 0$.

Key words: Hodge theory, Laplace operator, harmonic forms.

Introduction

By Hodge theory (1930), for a closed p-form ω be harmonic in a closed Riemannian manifold (M, g) is necessary and sufficient that ω be co-closed.

$$\Delta_g \omega = 0 \Leftrightarrow d\omega = \delta\omega = 0.$$

We know, however, that $\delta = *_g d *_g$ and the condition “be co-closed” depends on the metric. This allows ask:

Given a closed p-form ω on a closed Riemannian manifold (M, g) , when does there exist a metric h on M in a such way that $\Delta_h \omega = 0$?

This question was answered completely for 1-forms by Calabi in 1969.

We present a sufficient condition for the solution in the case of p-forms in each open of some atlas over M .

Results and Discussion

We search for a metric

$$h = e^{\frac{2f}{n}} g$$

where $f : M \rightarrow \mathbb{R}$ is a C^∞ function.

Let $*_g$ the Hodge star on the metric g . For ω be co-closed on h -metric:

$$d *_h \omega = 0.$$

Then the condition to ω be co-closed is the existence of a function $f \in C^\infty(M)$ such that

$$df \wedge *_g \omega = -d *_g \omega \quad (1)$$

We rewrite it in terms of a matrix equation:

Using the multi-index notation, any p-form ω can be written as:

$$\omega = \sum_I \omega_I dx^I, \\ I = \{i_1 < \dots < i_p\}.$$

Let J be the multi-index complementary of I .

Then we define the vector S^J of $\mathbb{R}^n$ whose coordinates $j = 1, \dots, n$ are:

$$(S^J)_j := \begin{cases} (-1)^\alpha \omega_{(J-i_k)}, & j=i_k \\ 0 & , \quad j \text{ not in } J. \end{cases}$$

where α means the signal of permutation to rearrange J as an ascending multi-index.

Let A the matrix whose entries are S^J , and choose a column vector c whose components are $-\nabla \cdot S^J$.

Then we rewrite the PDE (1) as:

$$\nabla f^t \cdot A = c \quad (2)$$

Conclusions

Once the problem reduces to linear algebra, the sufficient condition for the existence of conformal metric is given by the invertibility of A .

But since A is invertible we get:

$$\nabla f^t = c \cdot A^{-1}$$

Then the solution f is obtained by integration in each open set of an atlas over M .

Acknowledgement

I would like to thank H. Sá Earp and L. Sperança. This project, range number 2014/06913-0, was financed by FAPESP.

¹ Alcibiades Rigas, Sandro B. Cristiano, Francesco Mercuri. Differential Forms and their Integrals..