Introduction to unidimensional discrete dynamical systems

Ricardo M. Martins (PQ), Yudi B. Kawamura (IC)

Abstract
In this work we consider unidimensional dynamical systems, that is, systems generated by iterating a function defined on the real line. The main results we prove is the Sarkovskii theorem (and some applications), that claims that if there is a 3-periodic orbit for some real continuous function f, then f also possesses k-periodic orbits, for every k. We also prove a generalization of this theorem, considering another ordering < in the set of natural numbers and proving that, with this new order, if there is a n-periodic orbit for f and m>n then there is also a m-periodic orbit for f.

Key words: dynamical systems, chaos, quadratic map

Introduction

Iterating a function is the simplest way to generate a dynamical system. In this case, an orbit is the set of the iterates of a given point. The period of an orbit is its cardinality, if finite. We have three options: the orbit of a point can be a unitary set, and in this case we say that the point is a fixed point; the orbit can be a finite set with k elements, and in this case we say that the point is k-periodic; or the orbit can be an infinite set.

The periodic and fixed points are very important to describe the dynamics of a given function f.

In this work we study techniques to find k-periodic points. In particular, we prove a famous theorem that assure that if there is a 3-periodic point for f, then there are k-periodic points for f, for every other k.

The study of unidimensional discrete dynamics is the first step to start a more complete study of general dynamical systems, including continuous-time dynamical systems generated by flows of differential equations.

Results and Discussion

Observe that to calculate the k-periodic points of a given function f, we have to solve the equations $f^k(x)=x$, $f^{k-1}(x)\neq x$. Note that even in the case f is a quadratic polynomial, the first equation is a 2k-degree polynomial, so it is hard to explicit find its zeroes.

Also note that the k-periodic points of f are just the fixed points of $f^k$ that are not fixed points for $f^{k-1}$.

The next theorem is widely known as “periodic three implies chaos”:

Theorem (Sarkovskii) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. If there is a 3-periodic point for f, then there are k-periodic points for f, for every f.

We proved a generalization of the above theorem. Consider in the set of natural number the following ordering, where the dot means product.

$3 < 5 < 7 < \ldots < 2.3 < 2.5 < 2.7 < \ldots < 2^k.3 < 2^k.5 < 2^k.7 < \ldots < 4 < 2 < 1$

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. If $k<l$ (using the above defined order) and there is a k-periodic point for f then there is a l-periodic point for f.

The prove of these theorems involves just the intermediate value theorem and a lot of considerations about inverse images of intervals.

Conclusions

The Sarkovskii theorem is a powerful theorem in discrete dynamical systems, and it is very useful in applications. Its proof is very constructive, so it is possible to find k-periodic points, for big k, just by knowing that there is a s-periodic point, for small s<k, without solving non-linear equations.

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