Revisiting the Schwarzschild and Droste-Hilbert Solutions of Einstein Equations and the Kruskal-Szekeres Spacetime

Igor S. Mol Bessa (IC)

Abstract
Using the formalism of warped geometry, we scrutinize the differences between the Schwarzschild and the Droste-Hilbert solutions of Einstein equations. It is shown beyond any doubt that those solutions are different because the topologies of the manifolds corresponding to them are different. We then proceed to a mathematically rigorous study of the Kruskal-Szekeres spacetime, which provides the maximal extension of the Droste-Hilbert solution, and discuss the alternative approach followed by Kasner and Fronsdal which consists in embedding the Droste-Hilbert solution in a 6-dimensional vectorial manifold in order to derive its maximal extension.

Key words: Schwarzschild, Hilbert-Droste, General Relativity.

Introduction
The journal General Relativity and Gravitation (GRG) reprinted in 2003 the famous paper in which Schwarzschild (1916) consecrated himself as the first person to find an exact solution of the Einstein equations. Following the same volume of that journal, S. Antoci and D.-E. Liebscher published an editorial note claiming that the solution presented by Schwarzschild, describing the gravitational field generated by a point of mass, is not equivalent to the one currently taught in textbooks on General Relativity, even if still carrying his name. The latter being a solution which was, however, found by J. Droste and D. Hilbert just a year (1917) after Schwarzschild publication. This event culminated in a series of papers concerned with the equivalence and the nature of these two solutions.

Results and Discussion
Three years after the publication of the editorial of Antoci and Liebscher, the GRG released a rectification note claiming that the solutions of Schwarzschild and Droste-Hilbert must be equivalent given that there exists a coordinate transformation for which the metric found originally by Schwarzschild can be written in the same coordinate form as the one found by Hilbert and Droste.

However, the author of that note ignored that a spacetime is not defined only by a metric, but also and equally importantly by the topology of its corresponding manifold. And as a matter of fact, the Schwarzschild solution is homeomorphic to

\[ \mathbb{R} \times ]0, \infty[ \times S^2, \]

which physically corresponds to the topology of the Minkowski spacetime with the worldline of the gravitational source removed, having consequently no black hole and dispensing any procedure of maximal extension. On the other hand, the manifold of the Droste-Hilbert solution is found to possess the exotic disconnected topology of

\[ \mathbb{R} \times \{[0, \infty[ \times \{-2M\} \times S^2, \]

being consequently a different solution of the Einstein equations.

Conclusions

In our work, we derived and compared in details the solutions of Schwarzschild and Droste-Hilbert of Einstein equations, which describes the gravitational field of a mass point at rest. We concluded that these are actually different solutions of Einstein equations because their spacetimes posses distinct topologies. Such a difference, we stress, is not of secondary importance, as it is possible in principle to ascertain which one corresponds to the physical reality by verifying the existence of a black hole in the gravitational field produced by a mass point.


