PROPAGATION OF ACOUSTIC WAVES USING THE PARABOLIZED STABILITY EQUATIONS

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ABSTRACT

The Parabolized Stability Equations (PSE) constitute a technique in which a normal-mode Ansatz is considered for the solution of the linearized Navier-Stokes Equations. The Ansatz is decomposed into a rapidly and a slowly varying part in the direction of the mean-flow. While PSE have long been used for calculating disturbances in boundary layers focusing on the determination of hydrodynamic stability, their application to ducts is very restricted. This work aims at demonstrating the capabilities of PSE to compute the propagation of acoustic waves in ducts. PSE is an interesting tool since it can provide accurate results with a low computational cost compared to standard computational methods, thanks to the use of an Ansatz from the method of multiple scales that allows parabolization of the linearized equations. The focus is given to bypass ducts of turbofans where there is propagation of disturbances generated by the fan. PSE proves its value when axial variation of properties is present. In this work, the wall impedance and the mean-flow temperature were chosen to be varied along the cylindrical duct axis. These variables are defined, respectively, by the presence of acoustic liners and cooling/heating processes on the flow. Contours and axial distributions of PSE pressure fluctuations for the cases with axial gradient of temperature were compared with analytical results and showed good agreement. For varying impedance, an analytical solution is not available; however, the contours and axial distributions of PSE pressure fluctuations were in accordance with expected qualitative trends. For all cases tested, the real part of the axial wavenumber obtained with the PSE was in close agreement with the analytical, locally-parallel result, which can be obtained using classical acoustics. Hence, PSE is able to correctly describe the pressure disturbances field in cylindrical ducts with axially-varying impedance and temperature, with a reduced computational cost.

Keywords: duct acoustics, linear stability analysis, PSE.

1. INTRODUCTION

The problem of propagation of acoustic waves in ducts appears in a plethora of important applications. In some cases, as in ducts with slowly-varying cross-section and with temperature gradients, analytical solutions for the pressure fluctuations can be obtained [1–3]. Bypass ducts of turbofans are examples in which acoustic waves generated by the fan propagate in a slowly-varying channel. The importance of studying this problem relies on the fact that the fan acoustic...
radiation propagates and is radiated from the open end of the duct contributing to the farfield noise [4, 5]. Calculation is particularly relevant when the duct walls are covered with acoustic liners, which should be designed so as to damp propagating modes; thus accurate prediction of duct acoustics is relevant for the design of efficient liners. Moreover, interesting phenomena as trapped acoustic modes occur in the bypass duct and can lead to fan instability [6].

While widely used for computing wavepackets in mixing-layers and jets [7–11], Parabolized Stability Equation (PSE) face scarce application when the problem turns to propagation of acoustic waves in ducts. The present work aims at demonstrating the suitability of PSE to calculate this problem. The study is restricted to ducts with no mean-flow, focusing on cases with varying impedance on duct walls and of temperature within the duct; extension of the technique to calculate propagation with a mean-flow is straightforward. Whenever possible, results of the method are compared to analytical results from the literature.

2. THEORETICAL FRAMEWORK

We present in this section the PSE formulation in our method. We also present the analytical solution of propagation of sound in ducts with axially-varying temperature, which serves to validate the numerical results.

2.1 Parabolized Stability Equations

PSE is a derivative of Linear Stability Analysis in which the Navier-Stokes Equations are linearized around a base-flow and a normal-mode Ansatz is used. The feature that distinguish PSE is the use of an Ansatz presenting periodicity in time and azimuth (considering cylindrical coordinates). No assumption is made about the radial dependence of the perturbations. In the direction aligned with the flow \( x \), the disturbance is decomposed into a rapidly and a slowly varying part. Both the slowly-varying part \( \hat{q} \) and the axial wavenumber \( \alpha \) are allowed to vary in the streamwise direction [9]. PSE is able to model all phenomena satisfying the Ansatz for slowly-varying base flows consisting solely of modes with generalized group velocity in the direction where equations are parabolized. For acoustic problems, propagating and evanescent modes of arbitrary radial order satisfy this condition.

The PSE in cylindrical coordinates are presented in the following matrix system:

\[
[A(\bar{q}) + B(\bar{q}, \alpha, \omega)]\hat{q} + C(\bar{q}) \frac{\partial \hat{q}}{\partial x} + D(\bar{q}) \frac{\partial \hat{q}}{\partial r} = 0, \tag{1}
\]

which is solved for \( \hat{q} \) given by:

\[
\hat{q} = [\hat{u}_x, \ \hat{u}_r, \ \hat{u}_\theta, \ \hat{T}, \ \hat{\rho}]^T,
\]

where \( \hat{u}_x, \ \hat{u}_r, \ \hat{u}_\theta, \ \hat{T} \) and \( \hat{\rho} \) are the slowly-varying parts of the perturbations of axial, radial and azimuthal velocity, temperature and density, respectively. \( \bar{q} \) denotes the base-flow properties, which can be set considering a steady solution of the flow equation or a mean profile when turbulent flow is expected. Details of the derivation and the matrices \( A, B, C \) and \( D \) may be found elsewhere [12].
The complete perturbation $q$ is obtained by:

$$q(x,r,\theta,t) = \hat{q}(x,r)e^{i\int_{0}^{x} \alpha(\xi)d\xi}e^{im\theta}e^{-i\omega t}, \quad (2)$$

where $\alpha$ is computed by ensuring that $\hat{q}$ does not have an exponential dependence along $x$, leading to [9]:

$$\alpha(x) = \frac{\int_{0}^{r} -i|\hat{q}|^2 \frac{\partial \log \hat{q}}{\partial x} \eta d\eta}{\int_{0}^{r} |\hat{q}|^2 \eta d\eta}. \quad (3)$$

Equation (1) is subject to the impedance type boundary condition:

$$Z = \frac{\hat{P}}{\hat{u}_r}, \quad (4)$$

where $Z$ is the impedance, $\hat{P}$ is the pressure fluctuation and $\hat{u}_r$ is the radial velocity fluctuation. $\hat{P}$ can be linked with $\hat{T}$ and $\hat{p}$ by the non-dimensional linearized perfect gas relation, given as:

$$\hat{P} = \frac{\gamma - 1}{\gamma} (\hat{T} \hat{p} + \hat{p} \hat{T}), \quad (5)$$

where $\gamma$ is the ratio of specific heats, taken as 1.4 for propagation in air. For the complete pressure fluctuation $P$ (calculated with $q$), which is shown in the results, it suffices to drop the $\hat{\cdot}$ in the previous equation and consider $\rho$ and $T$ calculated with $q$.

The PSE is started with an eigensolution of the local problem at the inlet. Among the several modes resulting from the solution of the eigenproblem, it is chosen the plane wave mode for the azimuthal wavenumber $m = 0$. For the case with $m = 26$, it is selected the second radial wavenumber $\mu = 2$.

The variables are made non-dimensional as:

$$u = \frac{u^*}{a_{ref}^*}, \quad \rho = \frac{\rho^*}{\rho_{ref}}, \quad p = \frac{p^*}{\rho_{ref} a_{ref}^*}, \quad T = \frac{C_p T^*}{a_{ref}^*}, \quad (6)$$

where $^*$ indicates dimensional values, $a_{ref}^*$ is a reference value for the speed of sound and $C_p$ is the specific heat at constant pressure. Lengths are made non-dimensional by a reference duct radius $r_{ref}$.

The non-dimensionalized base-flow used consisted in a field with zero velocity ($u_x = u_r = u_\theta = 0$) and with density variation relied with the temperature distribution by:
where \( T(x) \) is the base-flow non-dimensionalized temperature distribution. The non-dimensionalized speed of sound is given by:

\[
\bar{a}(x) = \sqrt{(\gamma - 1)T(x)}.
\] (8)

The dimensional relation between the real part of the axial wavenumber \( \alpha^* (\alpha^*_r) \) and the phase velocity \( U^*_c \) is given by:

\[
\alpha^*_r = \frac{\omega}{U^*_c}.
\] (9)

For plane waves propagating inside the duct, the phase velocity is equal to the speed of sound, and the non-dimensional version of equation (9) leads to an expected locally-parallel result for the wavenumber at a station \( x \), given as:

\[
\alpha_r(x) = \frac{He}{\bar{a}(x)},
\] (10)

where \( He = \frac{\omega_{ref}R_{ref}}{a_{ref}} \) is the Helmoltz number.

### 2.2 Webster’s horn equation for axially-variable mean-flow temperature

The problem of low-frequency sound propagation in ducts with slowly varying temperature in the axial direction can be modeled by a special form of the Webster’s horn equation. Rienstra [13] provided a detailed derivation of this equation, given as:

\[
\psi'' + \left( \frac{He^2}{\bar{a}^2(x)} - \frac{d''(x)}{d(x)} \right) \psi = 0,
\] (11)

where \( \psi = P(x)d(x) \), \( d(x) = \sqrt{\pi}r(x)\bar{a}(x) \). \( r(x) \) is the function describing the radius of the duct.

### 3. TEST CASES

A cylindrical duct with radius \( r = 1 \) and no mean-flow was considered. The inlet and the outlet positions are \( x = 0 \) and \( x = 4 \), respectively. It was selected the impedance and the temperature as axially-variable parameters and a linear distribution between the properties at the inlet (\( Z_{in} \) and \( T_{in} \)) and the outlet (\( Z_{out} \) and \( T_{out} \)) was considered. The azimuthal wavenumber \( m = 0 \) was selected, since an analytical solution for acoustic propagation is available for this case. For \( m = 0 \),
it was considered a radial wavenumber corresponding to the propagation of nearly plane waves and $He$ was fixed to 40. $m = 26$ was also employed, because it represents the model problem of a fan with 26 blades, which was previously studied [1]. For $m = 26$, it was selected the second radial wavenumber $\mu = 2$ and $He = 200$. Since an analytical solution is not available for $m \neq 0$, only a qualitative analysis is provided for $m = 26$. Results for $m < 26$ are more accurate than those for $m = 26$ and are not shown. Table 1 summarizes the parameters used in the test cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>Azimuthal wavenumber ($m$)</th>
<th>Radial wavenumber ($\mu$)</th>
<th>Helmholtz number ($He$)</th>
<th>Impedance at inlet ($Z_{in}$)</th>
<th>Impedance at outlet ($Z_{out}$)</th>
<th>Temperature at inlet ($T_{in}$)</th>
<th>Temperature at outlet ($T_{out}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>plane wave</td>
<td>40</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>2.5</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>plane wave</td>
<td>40</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>2.5</td>
<td>6.0</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>2</td>
<td>200</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>2.5</td>
<td>6.0</td>
</tr>
<tr>
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<td>0</td>
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<td>40</td>
<td>100</td>
<td>0</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>plane wave</td>
<td>40</td>
<td>100</td>
<td>0</td>
<td>2.5</td>
<td>2.5</td>
</tr>
</tbody>
</table>

4. RESULTS

In this section, PSE and analytical contours of the real part of pressure perturbations $Re(P)$ are presented. Moreover, $Re(P)$ along a line in $x$ at $r = 0.9$ and the analytical evolution in $x$ of the real part of the axial wavenumber $\alpha_r$ (phase velocity) obtained with the PSE and analytically are compared. Note that for variable impedance, analytical results are not available.

4.1 Ducts with temperature distribution

The contours of $Re(P)$ obtained with PSE for $T_{in} = 2.5$ and $T_{out} = 6$ (positive axial gradient of temperature) are shown in figure 1 and those obtained analytically in figure 2. Comparisons between analytical and numerical $Re(P)$ along $x$ for $r = 0.9$ and $\alpha_r$ are presented in figures 3 and 4, respectively.

Figure 1: PSE contours of pressure fluctuations for $T_{in} = 2.5$ and $T_{out} = 6.0$, $m = 0$.

Figure 2: Analytical contours of pressure fluctuations for $T_{in} = 2.5$ and $T_{out} = 6.0$, $m = 0$.
PSE contours of pressure distributions (figure 1) indicates a plane wave which propagates without changing its shape. However, it is possible to notice that as $T$ increases, so does the wavelength $\lambda_x$. These contours are in close agreement with the analytical results shown by figure 2. The relationship between $\lambda_x$ and $T$ is given as:

$$\lambda_x = \frac{2\pi}{He [\overline{T}(x)]},$$

(12)

where $\overline{T}(x)$ is given by equation (8). It is possible to infer from equation (12) that the increase in temperature leads to an rise in $\lambda_x$ since $\overline{T}(x)$ is an increasing function.

The comparison between PSE and analytical results presented in figure 3 validates the numerical method. We observe a slight amplitude decrease as the wave propagates in the duct, and PSE reproduces accurately the analytical result.

Figure 4 shows the comparison between the numerical and analytical real part of the axial wavenumber. A nearly exact match is obtained. The plot shows the decrease of $\alpha_r$ with $x$ which is consistent with the increase of $\lambda_x$ since $\overline{T}(x)$ is an increasing function.

It was also considered a duct with axially-decreasing temperature. In this case, $T_{in} = 2.5$ and $T_{out} = 1.0$. The PSE and analytical contours of $Re(P)$ are shown in figures 5 and 6. Comparisons between the numerical and analytical $Re(P)$ along a line in $x$ and $\alpha_r$ are presented in figures 7 and 8, respectively.
In agreement with the previous analysis, Figure 5 indicates that the decrease in temperature with $x$ causes a reduction of the wavelength. This result can also be seen in the analytical contours shown by figure 6. The PSE results match closely the analytical contours.

Figure 7 provides a better quantitative comparison between the PSE and analytical fluctuations. The numerical results are in very good agreement with the solution obtained via Webster’s horn equation. Both shows the decrease of the wavelength and the increase of the amplitude of pressure disturbances with $x$. The PSE and analytical $\alpha_r$ also show good agreement as can be noticed in figure 8. These results indicate the trend of growth of $\alpha_r$ with the drop of temperature.

Finally, we present the results for $T_{in} = 2.5$ and $T_{out} = 6.0$, but now considering $m = 26$. For this case, an analytical solution for the pressure perturbations is not available. The PSE contours of pressure disturbances and the distribution of this variable along $x$ are shown in figures 9 and 10, respectively. Numerical and analytical results of $\alpha_r$ in function of $x$ are shown in figure 11.
The contours in figure 9 show that the pressure disturbances are no longer constant along the radius for a given \( x \), which is consistent with the choice of \( m = 26 \) and \( \mu = 2 \). It is seen that the pressure vanishes on the duct inner region, as expected for non-axisymmetric disturbances. In parallel with the contours for \( m = 0 \) (figure 1), there is an slight increase in the wavelength with \( x \). Moreover, as indicated by figure 10, the amplitude of the pressure disturbances decays with \( x \), similar to the obtained for \( m = 0 \).

An analytical solution for \( \alpha_r \) considering \( m > 0 \) is not available is this paper. However, we plotted in figure 11 the analytical result of \( \alpha_r \) for \( m = 0 \) with the numerical result for \( m = 26 \). There are obviously differences between the curves but they are not very large (4.0% at \( x = 4 \)). The trend shown by the plot is similar to the case with \( m = 0 \) (figure 4) indicating a drop in the real part of the axial wavenumber with the increase of the temperature. However, in the current case \( (m = 26) \), the percentual drop between the inlet and outlet is larger. This can be explained by the fact that the disturbances with higher radial and azimuthal wavenumbers are more severely damped as they propagate.

### 4.2 Ducts with impedance distribution at the wall

The PSE contours and the distribution of \( \text{Re}(P) \) along \( x \) for \( Z_{\text{in}} = 100 \) and \( Z_{\text{out}} = 0 \) are presented in figures 12 and 13, respectively.

![Figure 12: Contours of the real part of PSE pressure fluctuations for \( Z_{\text{in}} = 100 \) and \( Z_{\text{out}} = 0 \). \( m = 0 \).](image)

![Figure 13: PSE pressure fluctuations distribution along \( x \) for \( r = 0.9 \), \( Z_{\text{in}} = 100 \) and \( Z_{\text{out}} = 0 \). \( m = 0 \).](image)

Figure 12 shows the propagation of a wave in \( x \) which is affected by the impedance in the vicinities of the wall. The effects on the wave are too mild to be seen in the majority of the domain. They turn more prominent after \( x = 3.5 \), where it is possible to infer a reduction of the amplitude of the perturbations and departures from the shape of a plane wave. This trend can be observed in figure 13 which show an increased attenuation after \( x = 3.0 \). As \( Z \) evolves from 100 to 0, the wall behaves close to a soft, pressure-release surface and absorbs part of the energy of the pressure fluctuations. Hence, the correct trend is obtained for \( \text{Re}(P) \).

The evolution of the PSE real and imaginary parts of \( \alpha \) can be observed in figures 14 and 15.

It is possible to infer from figure 14 that there is no substantial variation in the real part of \( \alpha \) to \( x = 3.5 \). This result is consistent with constant wavelength in the \( x \) direction \( \lambda_x \) as observed in figures 12 and 13, since \( \alpha_r = \frac{2\pi}{\lambda_x} \). After the mentioned point, a sharp drop in \( \alpha_r \) can be seen. The imaginary part of \( \alpha \) shown in figure 15 has its value constant and close to 0 throughout the first half of the domain. After \( x = 2.0 \), there is a fast growth of the imaginary part of \( \alpha \) which attenuates the perturbations. This is the expected behaviour of an acoustic liner which acts to reduce the amplitude of the sound waves.
The attention is now turned to the variation of the imaginary part of $Z$ along $x$. Figures 16 and 17 show the PSE contours and the distribution along $x$ of $\text{Re}(P)$ for $Z_{\text{in}} = 100i$ and $Z_{\text{out}} = 0$.

As in the previous case, figure 16 shows that the effects of impedance are restricted to the neighborhood of the wall and particularly to the end of the domain. The imaginary impedance acts on the phase of the wave. Therefore, instead of seeing attenuation of the disturbances after $x = 3.5$, we observe a shift on the phase of the wave. The amplitude of $\text{Re}(P)$ remains constant along the most part of the domain as can be seen in figure 17; this is consistent with purely imaginary impedance. A slight growth in the amplitude of $\text{Re}(P)$ is, however, noted next to the outlet.

Both the real and imaginary parts of the wavenumber $\alpha$ were seen to remain constant in $x$ and close to 40 and 0, respectively, and is not shown here for brevity.

5. CONCLUSIONS

This work assessed the capability of the PSE of computing the propagation of acoustic waves in a duct with axially varying properties, such as wall impedance and base-flow temperature, which are of practical significance. Real and imaginary impedances were considered, and also ducts with increasing and decreasing temperature in the axial direction. We have considered propagation in a quiescent fluid.

For all tested cases, the real part of the axial wavenumber calculated with the PSE was in close agreement with analytical results. Considering the cases for which an analytical solution for the pressure disturbances was available (cases with variable temperature and $m = 0$), very good agreement between PSE and analytical results was obtained. Results for other conditions displayed the expected qualitative trends.
Therefore, PSE is able to compute the propagation of acoustic waves in ducts with axially-varying temperature and impedance. It also presents low computational cost, with typical runs lasting minutes in a common desktop computer, without any sort of optimization of the numerical algorithm.

Current work is focused on calculating the propagation of acoustic waves in ducts with varying cross-section and with non-zero base-flow. PSE is applicable to these conditions, and extension of the present results to such cases is straightforward.

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REFERÊNCIAS


