

MANAGING REPAIRABLE SUPPLIES USING MARKOV CHAINS.

OSCAR DE ALMEIDA MACHADO

Academia Militar das Agulhas Negras
Rod. Presidente Dutra Km 306, Paraíso, Resende-RJ, 27534-970.
Instituto Tecnológico de Aeronáutica
Praça Marechal Eduardo Gomes, 50 - Vila das Acácias, São José dos Campos - SP, 12228-900.
Universidade Federal de São Paulo
Avenida Cesare Mansueto Giulio Lattes, 1201 - Jardim Coqueiro, São José dos Campos - SP,
12247-014
oscardmachado2014@gmail.com

ANDRÉ LUIZ GOULART MATOS

Academia Militar das Agulhas Negras
Rod. Presidente Dutra Km 306, Paraíso, Resende-RJ, 27534-970.
aluizgoulartmatos@gmail.com

MAURI APARECIDO DE OLIVEIRA

Instituto Tecnológico de Aeronáutica
Praça Marechal Eduardo Gomes, 50 - Vila das Acácias, São José dos Campos - SP, 12228-900.
mauri@ita.br

ABSTRACT

This paper aims to present a mathematical approach to managing repairable supplies by employing discrete-time Markov chains with finite state spaces. Repairable items, such as aircraft spare parts and industrial machinery, for example, are frequently used in the world due to their potential for reuse. However, determining the optimal turnover quantity for such items is a very complex task. This study proposes modeling the shortage variable, obtained from historical demand and return data, as a Markovian stochastic process. Using Gaussian smoothing, a transition matrix was constructed, and the state vector represents the system's state changes over time. Some simulations using real-world data demonstrate the model's accuracy and practicality. This approach is both mathematically intuitive and computationally efficient, offering a robust and efficient tool for inventory optimization in repairable supply chains. Suggestions for further research include exploring transition matrix's spectral gap analysis and the use of continuous-time Markov chains with variable probabilities.

KEYWORDS. Repairable Supplies, Discrete-time Markov Chains, Gaussian smoothing.

Paper topics: Logistics and Transportation, OR in Management and Production, Other Applications in OR.

1. Introduction

Supply chain management (SCM) plays an important role in modern industrial and service sectors, ensuring the efficient flow of goods, services, and information across different stages of production and distribution. According to [Christopher 2016], effective SCM practices reduce operational costs and also improve clients' satisfaction and give a competitive advantage. Within this broad field, the management of repairable items is a particularly important area due to its potential for substantial cost savings and sustainability benefits.

Repairable items are components or products that, after experiencing failure, or another need of intervention, can be restored to a fully functional state instead of being discarded. Some examples include aircraft spare parts, medical equipment, military weapons, and industrial machinery like nuclear reactors, for example. For [Sherbrooke 2004], unlike consumable items that are replaced after use, repairable items undergo maintenance processes, such as inspection, repair, and testing, enabling their reuse in the operational cycle.

The repair of such items offers several advantages. The first one is the reduction of needs for frequent procurement of new components. The second is based on minimizing waste generation and lowering the environmental footprint of operations. From a financial perspective, focusing on repairable items can lead to significant cost savings. Furthermore, efficient repairable supply management can maintain the desirable service levels and avoid disruptions in the equipment's operations, reducing the lost profits of organization.

The major challenge in managing repairable inventories is determining the optimal quantity of items that should be kept in circulation to balance service levels and costs. If this quantity is overestimated, excess stock leads to unnecessary holding costs, while underestimating it can result in shortage costs and lost profits. To address this challenge, mathematical models, particularly Markov chains, have been effectively applied over time.

According to [Hillier and Liberman 2010], Markov chains model the probabilistic transitions of items between different states. By analyzing these transitions, it is possible to forecast the ideal turnover quantity required to maintain desired service levels while minimizing costs. This predictive capability enables decision-makers to plan more accurately and optimize the performance of repairable supply chains.

Based on the above, the main purpose of this work is applying Markov chains in the repairable supply chain management in order to determine the optimal turnover quantity for minimizing the logistical costs based on the ideal shortage probability.

2. Literature Review

According to [Konstantaras 2010] and [Konstantaras et al. 2010], the first model to address the repairable supplies management was developed by [Schrady 1967]. This model assumed constant demand and a constant material repair rate. In the next year, [Sheerbrooke 1968] developed a model based in multi-echelons of repair. This model assumed that the demand has Poisson distribution. In the same year, [Allen and d'Esopo 1968] proposed a model that determines the ordering policy, in a repairable supply chain, minimizing the total cost.

Several years later, [Nahmias and Rivera 1979] enhanced Schrady's original deterministic model. After a gap in publications on the topic in the 1980s, two significant contributions appeared: [Mabini et al. 1992], who extended Schrady's model to accommodate multiple items, and [Richter 1996], who developed inventory control policies based on unit purchase lots, reflecting the practice of acquiring high-cost repairable items individually.

Since the early 2000s, research in this field has expanded exponentially. [Koh et al. 2002] and [Teunter 2004] provide some advances in the models introduced by Schrady and by Nahmias and Rivera. Other contributions by [Wong et al. 2005], [Konstantaras et al. 2010], [Basten et al. 2012], [Tracht et al. 2013], [Arts 2016], and [Gehret et al. 2019] have focused on probabilistic models, offering alternatives to the earlier deterministic approaches. [Boujarif et al. 2024] presents

a multi-objective framework incorporating spare part reliability post-repair estimation into opportunistic maintenance decisions.

In Brazil, the subject is very little studied. Nationally, the works of [Igawa 2006], [Machado and Ferreira 2010], and [Machado 2017] given notables contributions about this subject. In the last Brazilian Symposium of Operations Research, we can found the work of [Machado et al. 2024] that proposed a simple framework to manage repairable supplies based on R-Studio software.

Regarding Markov Chains, the mathematical theory was first formulated by the Russian mathematician Andrei A. Markov (1856–1922) and was published in 1906. Since then, it has found numerous applications in everyday life, including in supply management. [Schaefer 1984] seems to be the first to use Markov chains in the development of repairable supply policies. [Selçuk 2013] employed a two-dimensional continuous-time Markov chain, which was explicitly solved using matrix geometric methods. More recently, [Gerrits et. al 2022] and [Hekimoğlu et. al 2022] used a continuous time Markov chain for performance evaluation having an Erlang distribution for the repair process. There are some examples of Markov chains application in Supply Chain Management.

3. Theoretical Basis

In this section, some concepts related to stochastic processes and Markov chains will be revisited, as they are essential for a full understanding of this work.

3.1. Stochastic Process and Markov Chains Definition

According to [Hillier and Lieberman, 2010], a stochastic process is an indexed set of random variables (X_n), where the index n runs over a given set N . Stochastic processes describe the behavior of a system operating over some period of time. In this way, variable X is observed to assume a certain value at a given time n .

The values that the variable X can assume at times n are called states. That is, if we assume that X is a discrete variable that can take any integer value from 0 to m , the system will have $m+1$ states, where the set $M = \{0, 1, 2, 3, 4, \dots, m\}$ represents the set of possible states for the system. Thus, the variable X at time $n = 1$ may assume the value 0, at time $n = 2$ the value 3, and at time $n = 3$ it may again assume the value 0. In other words, a stochastic process is characterized by the ability to define the value of the variable $X \in M$ at a given time $n \in N$.

If the set of values that time can assume is discrete (integers numbers), the process is referred to as a discrete-time stochastic process. Conversely, if time can take on any value within a continuous range, it is characterized as a continuous-time stochastic process. Moreover, if the variable X is restricted to a finite set of possible values, the process is said to possess a finite state space. In the present study, it is assumed that the stochastic process under analysis is a discrete-time stochastic process with a finite state space, that is, the set M is finite, and the set N consists exclusively of natural numbers, defined as $N = \{0, 1, 2, 3, \dots\}$.

Also according to [Hillier and Lieberman 2010], a stochastic process is said to possess the Markov property if the probability of any future event depends solely on the current event; in other words, the probability that the system will assume a given state at time $n+1$ depends only on its state at time n . Markov chains are thus stochastic processes that satisfy the Markov property. In summary, it can be stated that the probability of the system assuming a state j at time $n+1$ depends exclusively on the state i occupied at time n . This set of conditional probabilities, $P(X_{n+1} = j | X_n = i)$, $\forall n = 1, 2, 3, \dots$, can be interpreted as the probability of transitioning from state i to state j during the passage from time n to $n+1$.

Obviously, the transition probability between states can vary over time; that is, the probability of transitioning from state i to state j from time n to $n+1$ may differ from the transition between the same states i and j at a different time, such as from $n+1$ to $n+2$. If the transition probabilities remain the same over time, the process is said to have stationary transitions.

At this point, the importance of studying Markov chains becomes evident, as it enables the determination of a probability distribution over time n for the possible states of the system, knowing only the initial state of the system and its transition probabilities. In other words, it becomes possible to assess the probability that the system will assume a state i at any given time n , based solely on the system's initial state and the transition probabilities.

3.2. Transition Matrix and State Vector

The transition matrix, also called the stochastic matrix, is the matrix that contains all the transition probabilities between the possible states of the system. Thus, since the system has $m+1$ states, this is a square matrix of order $m+1$. A transition matrix (T) can be represented as follows:

$$T = \begin{matrix} & \begin{matrix} \text{State} & 0 & 1 & \dots & m \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \dots \\ m \end{matrix} & \begin{pmatrix} p_{00} & p_{01} & \dots & p_{0m} \\ p_{10} & p_{11} & \dots & p_{1m} \\ \dots & \dots & \ddots & \dots \\ p_{m0} & p_{m1} & \dots & p_{mm} \end{pmatrix} \end{matrix} \quad (1)$$

Where p_{ij} denotes the transition probability from state i to state j .

Note that Kolmogorov's axiom of the summation of probabilities still holds, meaning that the sum of the elements in each row of the transition matrix is equal to 1. Mathematically, this means that:

$$\sum_{j=0}^m p_{ij} = 1, \forall i \in M \quad (2)$$

The state vector (V) is the matrix that contains $m+1$ columns and only one row, whose elements indicate the probabilities that the system is in one of the $m+1$ states at a given time. The initial state vector, which we will denote by V_0 , will have one element equal to 1, and all other elements will be equal to 0. It is important to note that the state vector at time $n+1$ is equal to the state vector at time n multiplied by the transition matrix T . That is:

$$V_{n+1} = V_n T \quad (3)$$

For the case $n = 0$, we have:

$$V_1 = V_0 T \quad (4)$$

Since the probabilities to be worked with in the present study are stationary, we have that:

$$V_2 = V_1 T = V_0 T \cdot T = V_0 T^2 \quad (5)$$

$$V_{n+2} = V_{n+1} T = V_n T \cdot T = V_n T^2 \quad (6)$$

$$V_{n+b} = V_n T^b \quad (7)$$

It is worth noting that, because the state vector represents the set of probabilities that the system is in a given state, the property (2) remains always valid over time.

3.3 Transient, recurrent, and absorbing states

Finally, it is necessary to introduce the concepts of recurrent state, transient state, and absorbing state. According to [Hillier and Lieberman 2010], a transient state is one where, once reached, there is a possibility that the process will never return to that state again. In other words,

the state is visited a finite number of times until the system reaches its own equilibrium in another state.

On the other hand, a recurrent state is one where; after entering that state, the process can certainly return to it again. In summary, a state is recurrent if it is not transient. An absorbing state is one where, once the system enters it, it is never left again. That is, the probability of reaching an absorbing state from itself is always equal to 1. In mathematical terms, if state a is an absorbing state, the probability of reaching state a from itself (p_{aa}) is equal to 1.

4. Method

To model the problem in question, it is necessary to collect demand data by discretizing time into monthly intervals. To ensure that the sample is significant and statistically relevant, data from at least the last 20 (twenty) months should be collected. Such sample size is recommended in many reference books of Statistics, such as [Spiegel 1977], [Sweeney et al. 2017] and [Devore 2019], for example.

According to the same criterion, data related to the return of materials should be collected on a monthly basis, including materials that are newly acquired. It is assumed that a repaired repairable item performs the same function as a new item. The next step will be to combine these two variables (demand and return of items) into a third variable, which will be called shortage. The shortage variable is then provided by formula (8), as presented in [Machado and Ferreira 2010]:

$$f_n = f_{n-1} + D_n - Tr_n, \text{ and for } n=1 \quad f_1 = D_1 - Tr_1; \quad (8)$$

Where: f_n : Shortage in the n -th month;

D_n : Demand in the n -th month; and

Tr_n : Quantity of items returned in the n -th month.

At this point, it is important to note that the shortage variable is discrete, as it represents the number of items that must be supplied through the repairable supply cycle. Therefore, we can consider the set of possible values assumed by this variable as the possible set of system states. In other words, it is assumed that the system states are the possible values of the shortage variable.

As a simplification of the model, it is assumed that we are working with Markov Chains with finite states, so an interval must be chosen for the shortage variable data that covers practically 100% (in other words, the entirety) of the possible values of the shortage variable. A possible interval would be to take the mean as a base and set a range of 6 (six) standard deviations. It is worth noting that the shortage variable is generally greater than 0, since it does not make sense for a system to have a return of items greater than the accumulated demand, considering the system is balanced, that is, for each item that is scrapped, a new item is acquired for replacement.

Thus, the interval to be considered for constructing the transition matrix will be $[0, \text{int}(\mu + 6\sigma)]$, where μ is the mean of the shortage variable distribution, σ is the standard deviation of the shortage variable, and $\text{int}()$ denotes rounding up to the nearest integer number. The adoption of the value 6σ for the interval is due to the fact that, according to [Lawrence and Pasternak 2002], for many distributions (particularly smooth unimodal distributions), the range of values from the highest to the lowest covers roughly six standard deviations.

Thus, the transition matrix (T') for the model developed will be a square matrix of order $\mu + 6\sigma + 1$, and take the following form:

$$T' = \begin{matrix} & \text{States} & 0 & 1 & \dots & \mu + 6\sigma \\ \begin{matrix} 0 \\ 1 \\ \dots \\ \mu + 6\sigma \end{matrix} & \begin{pmatrix} p_{00} & p_{01} & \dots & p_{0 \mu+6\sigma} \\ p_{10} & p_{11} & \dots & p_{1 \mu+6\sigma} \\ \dots & \dots & \ddots & \dots \\ p_{\mu+6\sigma 0} & p_{\mu+6\sigma 1} & \dots & p_{\mu+6\sigma \mu+6\sigma} \end{pmatrix} \end{matrix} \quad (9)$$

After defining the interval that will contain the system states, it is necessary to determine the state transition probabilities. We used a Gaussian smoothing according to the formula (10), which provides the weights of the transition probabilities from state i to state j .

$$W(i, j) = e^{-\frac{(j-\mu)^2}{2\sigma^2}} \cdot e^{-\frac{(j-i)^2}{2\beta^2}} \quad (10)$$

Where: $W(i, j)$: Weight of transition probability of state i to state j ;

μ : mean of shortage variable;

σ : standard deviation of shortage variable; and

β : standard deviation of variability of the shortage variable, given by the differences $f_n - f_{n-1}$.

This formula allows closer states to have a higher transition probability, while more distant states have a lower probability. The probability is also higher when both the origin and destination states are closer to the mean of the distribution. The first term (σ) favors destinations near the mean, while the second term (β) favors destinations that are not too far from the current state. In summary, short steps are more likely than long steps.

Once the weights are obtained, the rows of the matrix are standardized in order to present the sum of the elements in each row is equal to 1. Note that since we are dealing with probabilities, all the properties of the transition matrix must be satisfied, including all Kolmogorov's axioms. This concludes the construction of the transition matrix.

In order to build the transition matrix, a Python program was used, which the user enters the order of the state transition matrix (N) and the parameters (μ , σ , β). With these parameters and the matrix order, the program constructs the transition matrix according to the weights provided by formula (10) and standardize all rows.

Once the state transition matrix has been built, the initial state vector (V_0) is set as the current state of the system. The element in position $N+1$ of the vector is equal to 1, assuming the last observed value of the shortage variable was n , and all other elements are set to 0. For example, let us suppose that the last observed value of the shortage variable was 12 units. In this case, the 13th element of the vector is set to 1, and the others are set to 0.

Once the transition matrix (T') and the initial state vector (V_0) are both defined, the subsequent state probability vectors are determined using formula (7) described in Section 3.2. It is important to note that, over time, this state vector evolves until it reaches a stationary value named steady state distribution.

To determine the ideal inventory level (or turnover quantity) for a given shortage probability (P_s), it is sufficient to compute the cumulative sum of the probabilities in the state vector. When the cumulative probability reaches the value $1-P_s$, the position of the last summed element more one unit corresponds to the ideal inventory level for that point in the desired time.

For this purpose, a code written in Python was used. It demands the user to input the mean and standard deviation of the shortage variable and then requests the user to input the state transition matrix generated by the previous program. The user must also enter the time horizon to be simulated. The program will then output the state probability distribution over the selected time horizon.

For instance, let us apply this to a specific item. Suppose the demand and return data are as shown in Table (1). The shortage variable is then calculated according to formula (8), as well as the variability of shortage variable according to the differences $f_n - f_{n-1}$.

From Table 1, we already have the initial parameters to construct the state transition matrix, as well as to determine the number of states it will have. The observed parameters are $\mu = 13.5$, $\sigma = 7.967325$ e $\beta = 1.627969$. The number of states N that the transition matrix will have is given by $\text{int}(13.5 + 6 \cdot 7.967325 + 1)$. In other words, in this example, we have $N = 63$ states.

Month/Year	Demand	Return	Shortage	$f_n - f_{n-1}$
jan/07	1	0	1	-
feb/07	2	1	2	1
mar/07	1	0	3	1
apr/07	3	1	5	2
may/07	3	1	7	2
jun/07	1	1	7	0
jul/07	0	0	7	0
aug/07	1	0	8	1
sep/07	1	0	9	1
oct/07	1	0	10	1
nov/07	2	0	12	2
dec/07	0	3	9	-3
jan/08	1	0	10	1
feb/08	4	0	14	4
mar/08	3	0	17	3
apr/08	5	1	21	4
may/08	0	1	20	-1
jun/08	3	0	23	3
jul/08	0	1	22	-1
aug/08	1	0	23	1
setp/08	0	0	23	0
oct/08	1	0	24	1
nov/08	1	1	24	0
dec/08	0	1	23	-1
MEAN	1.4583333	0.5	13.5	0.95652
STD DEV.	1.382473104	0.722315119	7.967325	1.627969

Table 1 – Table with shortage variable and $f_n - f_{n-1}$. Source: Authors.

The transition matrix, of order 63 in this case, is then obtained by standardizing the weights in each row. The weights are provided by formula (10). In this example, the first elements of the matrix are shown in Figure (1), and a heatmap of the state transition probabilities of the matrix is presented in Figure (2).

By closely examining the produced heatmap, it can be observed that the state transition probabilities are concentrated around the main diagonal of the transition matrix. This is entirely realistic, since abrupt transitions are uncommon, given that the observed variations in the shortage variable, given by $f_n - f_{n-1}$, are not very large. For instance, we believe that the probability of transition from state 0 to state 63 is almost 0. However, the probability of transition from 10 to 11 is greater.

After constructing the transition matrix, simulations are carried out considering the possible time intervals. The initial state vector (V_0) is a sparse vector containing zeros in all positions except for the 24th position, since the last observed value of the shortage variable was 23. Therefore, the initial state of the system is 23.

0.316291	0.321435	0.220492	0.10209	0.031905	0.00673	0.000958	9.21E-05	5.97E-06	2.62E-07	7.73E-09
0.178692	0.264838	0.264939	0.178897	0.081536	0.025084	0.005209	0.00073	6.91E-05	4.41E-06	1.9E-07
0.078391	0.169437	0.247196	0.243425	0.161801	0.072592	0.021983	0.004493	0.00062	5.77E-05	3.63E-06
0.025237	0.079552	0.169259	0.243077	0.235628	0.15417	0.068087	0.020297	0.004084	0.000555	5.08E-05
0.005761	0.026485	0.082181	0.172119	0.243321	0.232177	0.149538	0.065009	0.019076	0.003778	0.000505
0.00092	0.006166	0.027902	0.085223	0.175701	0.244503	0.229659	0.145604	0.06231	0.017998	0.003509
0.000102	0.001	0.0066	0.029398	0.088388	0.179379	0.245718	0.227193	0.141789	0.059729	0.016983
7.92E-06	0.000113	0.001087	0.00706	0.030957	0.091621	0.183033	0.246805	0.224631	0.137999	0.057224
4.27E-07	8.88E-06	0.000125	0.00118	0.007548	0.032578	0.094912	0.186645	0.247741	0.221959	0.134227
1.6E-08	4.86E-07	9.94E-06	0.000137	0.001281	0.008064	0.034262	0.09826	0.190208	0.248525	0.219181
4.18E-10	1.85E-08	5.52E-07	1.11E-05	0.000151	0.00139	0.00861	0.036011	0.101662	0.193716	0.249153
7.61E-12	4.91E-10	2.14E-08	6.28E-07	1.25E-05	0.000167	0.001506	0.009188	0.037826	0.105115	0.197165
9.63E-14	9.06E-12	5.75E-10	2.46E-08	7.13E-07	1.39E-05	0.000183	0.001632	0.009798	0.039706	0.108616
8.49E-16	1.16E-13	1.08E-11	6.74E-10	2.84E-08	8.09E-07	1.56E-05	0.000202	0.001767	0.010441	0.041654
5.21E-18	1.04E-15	1.41E-13	1.28E-11	7.89E-10	3.27E-08	9.18E-07	1.74E-05	0.000222	0.001911	0.01112
2.22E-20	6.49E-18	1.28E-15	1.7E-13	1.52E-11	9.23E-10	3.77E-08	1.04E-06	1.94E-05	0.000244	0.002067
6.61E-23	2.81E-20	8.08E-18	1.57E-15	2.05E-13	1.81E-11	1.08E-09	4.34E-08	1.18E-06	2.16E-05	0.000267
1.37E-25	8.49E-23	3.56E-20	1.01E-17	1.92E-15	2.47E-13	2.15E-11	1.26E-09	4.99E-08	1.33E-06	2.41E-05
1.97E-28	1.78E-25	1.09E-22	4.49E-20	1.25E-17	2.35E-15	2.98E-13	2.55E-11	1.47E-09	5.74E-08	1.51E-06
1.98E-31	2.61E-28	2.32E-25	1.4E-22	5.67E-20	1.55E-17	2.87E-15	3.59E-13	3.02E-11	1.72E-09	6.59E-08
1.38E-34	2.66E-31	3.45E-28	3.03E-25	1.79E-22	7.16E-20	1.93E-17	3.51E-15	4.31E-13	3.58E-11	2E-09
6.71E-38	1.88E-34	3.57E-31	4.56E-28	3.94E-25	2.29E-22	9.02E-20	2.4E-17	4.29E-15	5.19E-13	4.24E-11
2.27E-41	9.29E-38	2.57E-34	4.79E-31	6.03E-28	5.12E-25	2.94E-22	1.14E-19	2.97E-17	5.24E-15	6.24E-13

Figure 1 – Some elements of transition matrix for the example. Source: Authors.

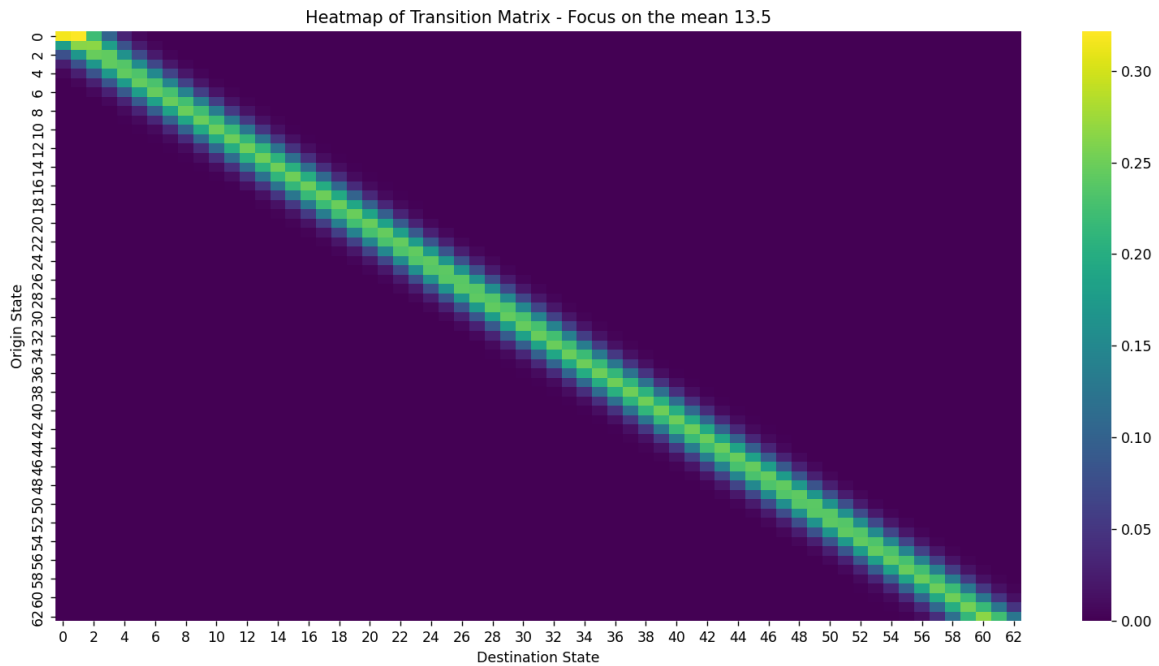


Figure 2 – Heatmap of transition matrix for the example. Source: Authors.

Time	t=1	t=2	t=5	t=10	t=20	t=30	t=40	t=50	t=100	t=1000
Quantity	24	25	25	25	24	23	22	21	20	20

Table 2 - Ideal turnover quantities observed in the simulated horizons. Source; Authors.

At this point, the program performs the sum of the first elements of the last obtained state vector until it reaches the value of $1-P_s$, where P_s is the ideal shortage probability. When the accumulated sum reaches the desired value, the position at which the algorithm stops corresponds to the ideal inventory level. For $t = 1$, and an ideal shortage probability of 0.1212, the last position

is given by the 25th element, meaning that 24 items should be in circulation at time $t = 1$. Running simulations for 2, 5, 10, 20, 30, 40, 50, 100, and 1000 months, we obtain the values shown in table (2). The Markov chain reaches a steady state at a certain time. In this case, the steady state is reached when $t = 226$.

Some simulations were carried out using data from 15 (fifteen) repairable items found in the [Machado and Ferreira, 2010], with the results described in the following section. The Python codes used to generate the transition matrix and run the simulations, as well as the resulting transition matrices and the collected data, are available at:

https://drive.google.com/drive/folders/1T9Hb4ItmaTvL3Dv2h_PNcHc8FhWGFyS?usp=sharing

5. Discussions and Results

The parameters used in the simulation are presented in table (3), where μ is the mean of shortage Variable, σ is the standard deviation of shortage variable, β is the standard deviation of variable $f_n - f_{n-1}$, N is the number of states of transition matrix, f_0 is the last observed value for the shortage variable, which will be the actual state of system and P_s is the ideal shortage probability calculated according the formula presented in [Machado 2024].

Item	μ	σ	β	N	f_0	P_s
1	13.5	7.967325	1.627969	63	23	0.1212
2	9.7917	5.191793	2.599998	42	9	0.1242
3	9.125	5.09315	1.258619	41	15	0.1428
4	9.8333	5.8135275	1.89317	46	16	0.1016
5	6.875	7.164602	2.376635	51	19	0.1212
6	11.6667	6.760928	1.668115	54	25	0.0919
7	11.625	8.218814	1.295623	62	26	0.1316
8	2.70833	0.907896	1.059657	10	3	0.1973
9	5.208333	2.7816036	1.727133	23	5	0.1692
10	16.7916667	7.598508	2.44949	64	24	0.1337
11	3.04166667	1.9219367	1.166644	16	4	0.1796
12	7.5416667	2.812691	1.266106	26	8	0.2312
13	4.2916667	2.92633	0.546513	23	8	0.2920
14	4.7083334	4.13342	1.209602	31	13	0.1480
15	2.70833	2.15647	0.89455	17	3	0.1849

Table 3 – Parameters used in the simulations. Source: Authors.

Simulations for the ideal turnover quantity were performed using time horizons of 1, 2, 5, 10, 20, 30, 40, 50, and 100 months. It is important to note that the Markov chain reaches its steady state at a certain point. Table (4) presents the simulations carried out for each of the 15 items, as well as the time horizon at which equilibrium is reached.

From the analysis of Table 4, it can be observed that some Markov Chains exhibit very fast convergence in just a few iterations, such as Item 8, which converged in only 14 iterations, and items 9 and 11, which required fewer than 40 iterations. Other chains converged much more slowly, taking over 200 iterations, as seen in Items 1, 7, and 13, for example.

At first glance, it was believed that the speed of convergence of the Markov Chain was due to the system having few states. This observation applies well to Items 1 and 7. However, Item 13 has only 23 states and converged only at $t = 287$.

Thus, it is observed that the speed of convergence of a Markov Chain is more related to its *spectral gap*, which is the difference between the largest and the second-largest eigenvalue of the transition matrix. At this point, a more detailed study of the eigenvectors and eigenvalues of

the transition matrices becomes necessary. However, this task is beyond the scope of this work but may serve as a direction for future research.

Item	t=1	t=2	t=5	t=10	t=20	t=30	t=40	t=50	t=100	Steady State
1	24	25	25	25	24	23	22	21	20	t=226
2	12	13	14	14	14	14	14	14	14	t=42
3	16	16	16	16	15	14	14	13	13	t=164
4	18	18	18	17	16	15	15	15	15	t=100
5	20	20	19	17	14	14	13	13	13	t=88
6	26	26	26	24	22	20	19	19	18	t=172
7	27	27	27	27	26	24	23	22	19	t=361
8	3	3	3	3	3	3	3	3	3	t=14
9	6	7	7	7	7	7	7	7	7	t=29
10	26	26	26	25	24	23	23	23	23	t=103
11	5	5	5	4	4	4	4	4	4	t=34
12	9	9	9	9	9	9	9	9	9	t=50
13	8	8	8	8	7	7	6	6	6	t=287
14	14	13	13	11	9	9	8	8	8	t=119
15	4	4	4	4	4	4	4	4	4	t=45

Table 4 – Ideal turnover quantities with the steady state observed. Source: Authors.

It is also observed that many items showed a fixed ideal turnover quantity across different time horizons in the simulations, as was the case for Items 8, 12, and 15. Although this occurred, the system only reached a stationary state at a later time, specifically, $t = 50$, and $t = 45$, respectively. From this, we can conclude that, in these cases, the changes in the state vector over time were not significant enough to alter the cumulative probability sum. Note that, the speed of convergence of the Markov Chain should not be confused with its intensity of convergence.

In this study, Gaussian smoothing was used to determine the state transition probabilities and to construct the stochastic matrix. This method proved to be quite efficient, yielding very plausible results considering the data used. As a suggestion for future work, it is proposed to explore other smoothing techniques or even to determine these transition probabilities experimentally using historical data. A longer historical series or one discretization into shorter intervals such as biweekly, weekly, or even daily, could also be used to obtain more observations.

Finally, it is worth noting that Markov Chains are a powerful tool for predicting the probabilities a system may present over a given time horizon. In this case, the use of Markov Chains enabled the prediction of ideal cycle quantities over a very long-time horizon (up to 100 months, for example). However, the transition matrix must always be updated to correct for possible deviations or externalities that were not captured in the historical data.

6. Conclusion

Based on all that has been presented, the importance of Markov Chains in forecasting the ideal turnover quantity for repairable supplies becomes evident. This tool brings a highly scientific character to the problem of sizing the cycle stock of repairable items, providing a solid mathematical basis for supply forecasting.

The proposed method is quite simple to implement, requiring only a computer capable of running Python, and the code used is compact and easy to understand. Mathematically, the method is also very intuitive, relying on basic concepts from matrix algebra, such as matrix multiplication. One of the main strengths of the method is that it does not require extensive or voluminous data, which are typically readily available in company and organizational databases.

As suggestions for future research, one could incorporate the determination of the *spectral gaps* of the state transition matrices to better understand the convergence behavior of the Markov Chain. It is also possible to explore the use of a Markov Chain with variable (non-stationary) transition probabilities or even a continuous-time Markov Chain to model the problem.

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