

## Role Assignments in some Circulant Graphs

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### ABSTRACT

An *r*-role assignment of a simple graph  $G$  is an assignment of  $r$  distinct roles to the vertices of  $G$ , such that two vertices with the same role have the same set of roles assigned to related vertices. Furthermore, a specific *r*-role assignment defines a *role graph*, in which the vertices are the distinct  $r$  roles, and there is an edge between two roles whenever there are two adjacent vertices in the graph  $G$  that correspond to these roles. This study focuses on circulant graphs, which are undirected graphs with cyclic symmetry with edges determined by a set of distances. We show necessary and sufficient conditions for the existence of *r*-role assignments in all circulant graphs of the form  $C_n(1)$ , for every fixed  $r \geq 2$ .

**KEYWORDS.** Circulant graphs, role assignment, graph theory

**Paper topics** (Role assignment, circulant graphs)

## 1. Introduction

The simplification of structures constitutes a significant aspect across diverse fields of science and engineering, including graph theory and artificial intelligence (AI). Graphs find important applications in social networks, distributed systems, recommendation systems, and graph neural networks. In this context, especially when using neural networks, simplifying graphs is very important to make models easier to understand and to reduce how much computing power they need. This helps with handling and analyzing large amounts of data.

One approach for this simplification can be a role assignment, which involves mapping the vertices of a graph to a smaller set of roles while preserving fundamental structural relationships, such as the neighborhood of each vertex [Lerner, 2005]. This concept was initially introduced by Angluin [1980] as a tool for processor networks and subsequently formalized by Everett and Borgatti [1991] under the designation of role coloring.

A *role assignment* (or *role coloring*) in a simple graph  $G$  is an assignment of  $r$  distinct roles to the vertices of  $G$  such that if two vertices share the same role, then the sets of roles of their neighbors are identical [Everett and Borgatti, 1991]. From such an assignment, a *role graph*  $R$  is derived, where the vertices represent the  $r$  distinct roles, and an edge exists between two roles if and only if there exist two adjacent vertices in  $G$  corresponding to these roles Lerner [2005]. The role graph does not contain multiple edges but permits loops, given that two adjacent vertices in  $G$  may be assigned the same role.

That concept finds applications in the modeling of social networks [Pekeč and Roberts, 2001] and in distributed computing [Chalopin et al., 2004, 2006]. Regarding to related work, it is known that deciding whether a graph  $G$  has a  $r$ -role assignment is polynomial-time solvable if  $r = 1$  and NP-complete for any fixed  $r \geq 2$  [Fiala and Paulusma, 2005], even for graphs with maximum degree 3 [Purcell and Rombach, 2021]. Furthermore, for any fixed  $r \geq 2$ , there exists a polynomial-time algorithm for computing a  $r$ -role assignment on trees, cographs [Purcell and Rombach, 2015], and proper interval graphs [Heggernes et al., 2012]. Particularly, Purcell and Rombach [2015] also characterized  $r$ -role assignment for path graphs:

**Theorem 1.1.** [Purcell and Rombach, 2015] *A path graph  $P_n$  has a  $r$ -role assignment if and only if  $n = (r - 1)k + r$  or  $n = (2r - 1)k + 2r$ , for some  $k \in \mathbb{N}$ .*

Up to our knowledge,  $r$ -role assignments in circulant graphs has not yet been studied and this work aims to initiate a systematic investigation on this direction. A *circulant* graph  $G = C_n(d_1, d_2, \dots, d_\ell)$  is a simple graph on  $n$  vertices, with vertex set  $V(G) = \{0, 1, \dots, n - 1\}$  and edge set  $E(G) = \bigcup_{i=1}^{\ell} \{(v, (v + d_i) \bmod n) \mid v \in V(G)\}$ . In other words, a vertex  $v$  is adjacent to vertices  $(v + d_i) \bmod n$  and  $(v - d_i) \bmod n$  for each  $d_i$  in the sequence, and  $d_i$  is called a *jump* or *generator* [Al-Labadi et al., 2023; Ilić and Bašić, 2010].

Motivated by Purcell and Rombach [2015] (Theorem 1.1), in this paper we present similar patterns of formulas that characterize all circulant graphs with one-generator admitting  $r$ -role assignments, for every fixed  $r \geq 2$ .

This work is structured as follows: Section 2 contains some basic definitions on graphs, role assignments and properties of circulant graphs; Sections 3, 4, and 5 addresses the problem in circulant graphs, with a focus on  $r$ -role assignments, for  $r = 2, 3$ , and  $r \geq 3$ , respectively. Although the results in Section 4 are generalized later in Section 5, that section was included to enhance clarity. Finally, Section 6 recalls the main findings and suggests directions for future research.

## 2. Preliminaries

We begin this section by presenting some basic definitions on graphs in Subsection 2.1. The definitions in the context of role assignments follow in Subsection 2.2.

## 2.1. Basic concepts on graphs

A graph  $G$  is a pair  $(V(G), E(G))$ , where  $V(G)$  is a set of vertices and  $E(G)$  is a set of edges. The vertices  $u$  and  $v$  are *adjacent* or *neighbors* if they are joined by an edge  $e = (u, v)$ ; we say that  $u, v$  are *incident* to  $(u, v)$ .

A *loop* is an edge incident to only one vertex. Two edges  $e_1$  and  $e_2$  are *parallel* if  $e_1 = e_2 = (u, v)$ . The *open neighborhood* of a vertex  $v$ , denoted by  $N_G(v)$ , is the set of all neighbors of  $v$  in  $G$ . The *closed neighborhood* of a vertex  $v$ , denoted by  $N_G[v]$ , is the set  $N_G(v) \cup \{v\}$ . When clear in the context, the subscript  $G$  will be omitted from the notation. A *simple graph* is a graph without loops and without parallel edges. In a simple graph  $G$ , the *degree* of a vertex  $v$  is the cardinality of  $N_G(v)$ . The *open neighborhood of a subset*  $U$  of  $V(G)$ , denoted as  $N_G(U)$ , is the union of the neighborhoods of the vertices of  $U$ . A vertex of degree zero is *isolated* and a *leaf* is a vertex of degree 1.

Two graphs  $G = (V(G), E(G))$  and  $G' = (V(G'), E(G'))$  are *isomorphic* if and only if there is a bijection, called *isomorphism function*,  $\phi : V(G) \rightarrow V(G')$  such that  $uv \in E(G)$  if and only if  $\phi(u)\phi(v) \in E(G')$ , for every  $u, v \in V(G)$ . We denote by  $G \simeq G'$  if  $G$  and  $G'$  are isomorphic and  $G \not\simeq G'$  otherwise.

A graph  $G$  is a *bipartite graph* if one can partition  $V(G) = A \cup B$  so that if there is an edge  $uv \in E(G)$ , then  $u \in A$  and  $v \in B$ , or vice versa. A graph is *k-regular* if every vertex in  $V(G)$  has degree  $k$ .

Concerning circulant graphs, we present some properties below. To avoid redundant representations, the jump distance  $d_i$  is considered such that  $d_i \leq \lfloor \frac{n}{2} \rfloor$ .

**Theorem 2.1** (Symmetry).  $C_n(d) \simeq C_n(n-d)$ .

*Proof.* Let  $n$  be the number of vertices of a circulant graph and  $d$  be the jump distance in  $C_n(d)$ , with  $d \leq \lfloor \frac{n}{2} \rfloor$ .

Starting from a vertex  $v$ , there is an edge connecting to  $(v+d) \bmod n$  in  $C_n(d)$ . Analogously, starting from  $(v+d) \bmod n$  in  $C_n(n-d)$ , there is an edge connecting to  $(v+d+(n-d)) \bmod n$ . This simplifies to  $(v+n) \bmod n = v$ .

Therefore, in  $C_n(d)$ , we have edges  $(v, v+d)$ , and in  $C_n(n-d)$ , we have edges  $(v+d, v)$ . Since the graphs are undirected, we have that  $C_n(d) \simeq C_n(n-d)$ .  $\square$

**Theorem 2.2** (Regularity). Let  $G = C_n(d_1, \dots, d_\ell)$  be a circulant graph with  $1 \leq d_1 < \dots < d_\ell \leq \lfloor n/2 \rfloor$ . If  $d_\ell \neq n/2$ , then  $G$  is  $2\ell$ -regular and if  $d_\ell = n/2$ , then  $G$  is  $(2\ell-1)$ -regular.

*Proof.* The neighbors of a vertex  $v$  in  $G$  are  $N(v) = \{v \pm d_i \pmod n \mid i = 1, \dots, \ell\}$ . The degree of  $v$  is  $|N(v)|$ . We will show that  $|N(v)| = 2\ell$ , by proving that the  $2\ell$  elements  $v+d_1, \dots, v+d_\ell, v-d_1, \dots, v-d_\ell$  are distinct modulo  $n$ .

We analyze the possible equalities (always modulo  $n$ ):

1.  $v+d_i \equiv v-d_i$ : This implies  $2d_i \equiv 0 \pmod n$ . Since  $1 \leq d_i \leq \lfloor n/2 \rfloor$ , this could only occur if  $2d_i = n$ , i.e.,  $d_i = n/2$ . By hypothesis,  $d_i \neq n/2$ , thus  $v+d_i \not\equiv v-d_i$ .
2.  $v+d_i \equiv v+d_j$  or  $v-d_i \equiv v-d_j$  (for  $i \neq j$ ): Both imply  $d_i \equiv d_j \pmod n$ . Since  $1 \leq d_i, d_j \leq \lfloor n/2 \rfloor < n$  and  $d_i \neq d_j$ , this is impossible.
3.  $v+d_i \equiv v-d_j$  (for any  $i, j$ ): This implies  $d_i + d_j \equiv 0 \pmod n$ . Since  $1 \leq d_i, d_j \leq \lfloor n/2 \rfloor$ , we have  $2 \leq d_i + d_j \leq 2\lfloor n/2 \rfloor \leq n$ . The equality  $d_i + d_j = n$  would require  $n$  to be even and  $d_i = d_j = n/2$ , but the hypothesis  $d_k \neq n/2$  prevents this. Thus,  $0 < d_i + d_j < n$ , making the congruence  $d_i + d_j \equiv 0 \pmod n$  impossible.

Since all  $2\ell$  potential neighbors  $v \pm d_i$  are distinct modulo  $n$ , the degree of  $v$  is exactly  $2\ell$ . As  $v$  is arbitrary,  $G$  is  $2\ell$ -regular.

When  $d_\ell = n/2$ , by the above  $C_n(d_1, \dots, d_{\ell-1})$  is  $2(\ell-1)$ -regular and the fact that  $C_n(n/2)$  is a disjoint union of  $n/2$  edges, that is, edges between  $i$  and  $i + n/2$ , for  $i = 0, \dots, n/2 - 1$ .  $\square$

**Theorem 2.3** (Decomposition into Cycles). *Let  $C_n(d)$  be a circulant graph with  $\gcd(n, d) = t$ . Then,  $C_n(d) \simeq C_{n/t}^t$ .*

*Proof.* Consider a circulant graph  $C_n(d)$  where  $\gcd(n, d) = t$ . The vertices can be identified with the elements of the ring  $\mathbb{Z}_n$ . Recall that each vertex  $v$  is linked to  $(v \pm d) \bmod n$ . We can partition the set of vertices  $V(C_n(d))$  into  $t$  disjoint classes,  $V_0, V_1, \dots, V_{t-1}$ , where each class  $V_r$  is given by  $V_r = \{v \in \mathbb{Z}_n \mid v \equiv r \pmod{t}\}$ . Each of these classes  $V_r$  represents a coset of  $\mathbb{Z}_n$  modulo  $t$  and contains exactly  $m = \frac{n}{t}$  elements, as inferred by modular arithmetic.

Since  $t$  divides  $d$ , we can write  $d = t \cdot d'$  for some integer  $d'$ . If we take a vertex  $v \in V_r$ , its neighbors are  $u = (v \pm d) \bmod n$ . Calculating  $u \bmod t$ , we have  $u \equiv (v \pm td') \bmod t \equiv v \bmod t \equiv r \pmod{t}$ . This shows that the neighbors of any vertex  $v \in V_r$  also belong to  $V_r$ . Consequently, there are no edges connecting vertices from distinct classes  $V_r$ . The graph  $C_n(d)$  thus decomposes into  $t$  disjoint components, each corresponding to a class  $V_r$ .

Within a specific class  $V_r$ , the elements can be written in the form  $v = r + kt$ , where  $0 \leq k \leq m-1$ , with  $m = \frac{n}{t}$ . Starting from a vertex  $r$  (corresponding to  $k = 0$ ) and repeatedly adding  $d$ , we generate the sequence  $r, r + d, r + 2d, \dots, r + (m-1)d \pmod{n}$ . Since  $d = td'$  and  $mt = n$ , this sequence traverses all  $m$  elements of  $V_r$  without repetition. After  $m$  additions of  $d$ , we obtain  $r + md = r + nd' \equiv r \pmod{n}$ , closing a cycle. Therefore, the vertices within each  $V_r$  form a cycle of length  $m$ .

Given that each class  $V_r$  induces a cycle  $C_m$  (with  $m = n/t$ ) and there are no edges between distinct classes, we conclude that the graph  $C_n(d)$  is isomorphic to the disjoint union of  $t$  copies of the cycle  $C_m$ . Formally,  $C_n(d) \simeq \bigcup_{r=0}^{t-1} C_m \simeq C_{n/t}^t$ .  $\square$

## 2.2. Fundamentals of Role Assignments

To formally define a role assignment, we recall the concept of a graph homomorphism. Consider  $G$  a simple graph and  $R$  a graph possibly containing loops. A *homomorphism* from  $G$  to  $R$  is a function  $p : V(G) \rightarrow V(R)$  such that, for every edge  $uv \in E(G)$ , the image  $p(u)p(v) \in E(R)$ .

When the homomorphism  $p$  is surjective and, additionally, the restriction of  $p$  to the neighborhood of each vertex  $u \in V(G)$  is also surjective – that is, the function  $p_u : N_G(u) \rightarrow N_R(p(u))$  is surjective – we say that  $p$  is a *locally surjective homomorphism* from  $G$  to  $R$ . In this context,  $p$  is called an  *$R$ -role assignment* of  $G$ . When  $r = |V(R)|$ , the assignment is called an  *$r$ -role assignment*.

For such a homomorphism to be possible, notice that  $n \geq 1$  and  $n \geq r$ , where  $n$  is the order of  $G$  and  $r$  is the order  $R$ . We present below a useful lemma that implies directly in the subsequent corollary.

**Lemma 2.4.** [Hell and Nešetřil, 2004] *Let  $G$  and  $H$  be two graphs. If  $p$  is a homomorphism from  $G$  to  $H$ , then the distance from  $p(u)$  to  $p(v)$  is at most the distance from  $u$  to  $v$  for any  $u, v \in V(G)$ .*

**Corollary 2.5.** *If  $G$  is a connected graph, then the role graph of every role assignment of  $G$  is also connected.*

### 3. Circulant Graphs with 2-Role Assignment

To determine if a graph  $G$  admits a 2-role assignment, one must examine the six possible role graphs with two vertices. However, given that circulant graphs  $C_n(1)$  are connected, by Corollary 2.5 we disregard the disconnected role graphs. Figure 1 illustrates the three connected role graphs considered. We assume, by symmetry, that in each role graph  $R_{2,i}$ , for  $i = 4, 5, 6$ , the black vertex is associated with role 1 and the white vertex with role 2, as depicted in Figure 1.

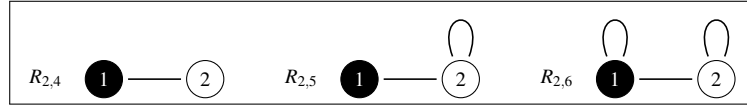


Figure 1: Possible connected role graphs resulting from a 2-role assignment.

As demonstrated by Roberts and Sheng [2001], for a graph  $G$  to admit an  $R_{2,4}$ -role assignment,  $G$  must be bipartite and contain no isolated vertices. This implies that  $C_n(1)$  has an  $R_{2,4}$ -role assignment if and only if  $n = 2k$  for some  $k \in \mathbb{N}^*$ .

Next, we investigate the conditions under which  $C_n(1)$  admits an  $R_{2,5}$ -assignment, leading to the following theorem:

**Theorem 3.1.** *The graph  $G = C_n(1)$  has an  $R_{2,5}$ -role assignment if and only if  $n = 3k$  for some  $k \in \mathbb{N}^*$ .*

*Proof.* ( $\Leftarrow$ ) Assume  $n = 3k$ . Define  $p : V(G) \rightarrow \{1, 2\}$  as follows:

$$p(v) = \begin{cases} 1 & \text{if } v \bmod 3 = 0, \\ 2 & \text{if } v \bmod 3 = 1 \text{ or } v \bmod 3 = 2. \end{cases}$$

Consider  $v - 1$  and  $v + 1$  always modulo  $n$ :

- If  $v \bmod 3 = 0$ , then  $p(v) = 1$ . Thus,  $p(N_G(v)) = p(\{v - 1, v + 1\}) = \{2\}$ , since  $(v - 1) \bmod 3 = 2$  and  $(v + 1) \bmod 3 = 1$ , which implies  $p(v - 1) = 2$  and  $p(v + 1) = 2$ . Therefore,  $p(N_G(v)) = \{2\}$  and  $N_R(1) = \{2\}$ .
- If  $v \bmod 3 = 1$ , then  $p(v) = 2$ . Thus,  $p(N_G(v)) = p(\{v - 1, v + 1\}) = \{1, 2\}$ , since  $(v - 1) \bmod 3 = 0$  and  $(v + 1) \bmod 3 = 2$ , which implies  $p(v - 1) = 1$  and  $p(v + 1) = 2$ . Therefore,  $p(N_G(v)) = \{1, 2\}$  and  $N_R(2) = \{1, 2\}$ .
- If  $v \bmod 3 = 2$ , then  $p(v) = 2$ . Thus,  $p(N_G(v)) = p(\{v - 1, v + 1\}) = \{1, 2\}$ , since  $(v - 1) \bmod 3 = 1$  and  $(v + 1) \bmod 3 = 0$ , which implies  $p(v - 1) = 2$  and  $p(v + 1) = 1$ . Therefore,  $p(N_G(v)) = \{1, 2\}$  and  $N_R(2) = \{1, 2\}$ .

( $\Rightarrow$ ) Assume that  $G = C_n(1)$  has an  $R_{2,5}$ -role assignment with  $N_R(1) = \{2\}$  and  $N_R(2) = \{1, 2\}$ . We may assume, without loss of generality, that  $p(0) = 1$ . Consequently,  $p(n - 1) = p(1) = 2$ .

We will show by induction on  $v$  that the following function  $p$  correctly maps the neighborhoods.

$$p(v) = \begin{cases} 1 & \text{if } v \bmod 3 = 0, \\ 2 & \text{if } v \bmod 3 = 1 \text{ or } v \bmod 3 = 2. \end{cases}$$

Assume this holds for all  $t$  such that  $0 \leq t \leq v$  for some  $v \geq 1$ , and show that  $p(v + 1) = 1$  if  $v + 1 \bmod 3 = 0$  and  $p(v + 1) = 2$ , otherwise.

- If  $(v+1) \bmod 3 = 0$ , then  $(v-1) \bmod 3 = 1$  and  $v \bmod 3 = 2$ . By the induction hypothesis,  $p(v) = 2$  and  $p(v-1) = 2$ . Since  $N_R(2) = \{1, 2\}$ , we conclude that  $p(v+1) = 1$ .
- If  $(v+1) \bmod 3 = 1$ , then  $(v-1) \bmod 3 = 2$  and  $v \bmod 3 = 0$ . By the induction hypothesis,  $p(v) = 1$  and  $p(v-1) = 2$ . Since  $N_R(1) = \{2\}$ , we conclude that  $p(v+1) = 2$ .
- If  $(v+1) \bmod 3 = 2$ , then  $(v-1) \bmod 3 = 0$  and  $v \bmod 3 = 1$ . By the induction hypothesis,  $p(v) = 2$  and  $p(v-1) = 1$ . Since  $N_R(2) = \{1, 2\}$ , we conclude that  $p(v+1) = 2$ .

Therefore, for all  $v$ , the function  $p$  holds. Now, we show that this mapping is valid only for  $n = 3k$ . We have that if  $p(n-1) = 2$ , then  $(n-1) \bmod 3 \neq 0$ . Since  $N_G(n-1) = \{0, n-2\}$  and  $p(0) = 1$ , we have  $p(n-2) = 2$ . Consequently,  $(n-2) \bmod 3 \neq 0$ . Thus,  $n \bmod 3 \neq 1$  and  $n \bmod 3 \neq 2$ , which implies  $n \bmod 3 = 0$ . This leads to the conclusion that  $\exists k \in \mathbb{N}^*$  such that  $n = 3k$ .  $\square$

In Theorem 3.1, we have seen that the  $R_{2,5}$ -role assignment on  $C_n(1)$  requires  $n$  to be a multiple of 3. The characterization for a  $R_{2,6}$ -role assignment (in this case,  $n = 4k$  for some  $k \in \mathbb{N}^*$ ) is analogous and thus omitted for brevity. Table 1 summarizes the conditions on  $n$  for 2-role assignments in  $C_n(1)$ .

Table 1: Conditions for 2-role assignments in circulant graphs  $C_n(1)$ .

| R-Role Assignment | Condition on $n$ |
|-------------------|------------------|
| $R_{2,4}$         | $n = 2k$         |
| $R_{2,5}$         | $n = 3k$         |
| $R_{2,6}$         | $n = 4k$         |

#### 4. Circulant Graphs with 3-Role Assignment

With respect to 3-role assignments, there are ten possible connected role graphs, cf. [Castonguay et al., 2025]. Since our focus is restricted to the circulant graph  $C_n(1)$  which is 2-regular, the possible role graphs are those with maximum degree at most 2, as can be seen in Figure 2.

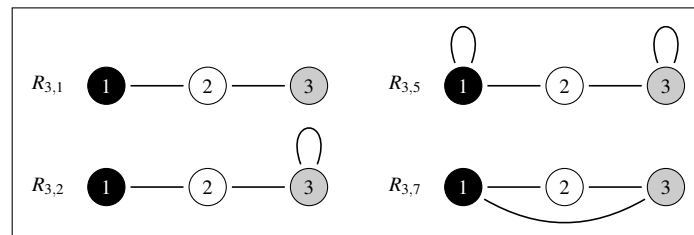


Figure 2: Possible connected role graphs arising from a 3-role assignment for  $C_n(1)$ .

In analyzing the properties of 3-role assignments, we examine the conditions under which  $C_n(1)$  admits an  $R_{3,1}$ -role assignment.

**Theorem 4.1.** *The graph  $G = C_n(1)$  admits an  $R_{3,1}$ -role assignment if and only if  $n = 4k$  for some  $k \in \mathbb{N}$ .*

*Proof.* ( $\Leftarrow$ ) Let  $n = 4k$ . We define  $p : V(G) \rightarrow \{1, 2, 3\}$  as follows:

$$p(v) = \begin{cases} 1 & \text{if } v \bmod 4 = 0, \\ 2 & \text{if } v \bmod 4 = 1 \text{ or } v \bmod 4 = 3, \\ 3 & \text{if } v \bmod 4 = 2. \end{cases}$$

Consider  $v-1$  and  $v+1$  always modulo  $n$ . We must prove that the mapping function  $p$  onto the graph  $C_n(1)$  with  $n = 4k$  preserves the neighborhood of  $R_{3,1}$ :

- If  $v \bmod 4 = 0$ , then  $p(v) = 1$ . Thus,  $p(N_G(v)) = p(\{v-1, v+1\}) = \{2\}$ , since  $(v-1) \bmod 4 = 3$  and  $(v+1) \bmod 4 = 1$ . Thus,  $p(v-1) = 2$  and  $p(v+1) = 2$ . Therefore,  $p(N_G(v)) = \{2\}$  and  $N_R(1) = \{2\}$ .
- If  $v \bmod 4 = 1$ , then  $p(v) = 2$ . Thus,  $p(N_G(v)) = p(\{v-1, v+1\}) = \{1, 3\}$ , since  $(v-1) \bmod 4 = 0$  and  $(v+1) \bmod 4 = 2$ . Thus,  $p(v-1) = 1$  and  $p(v+1) = 3$ . Therefore,  $p(N_G(v)) = \{1, 3\}$  and  $N_R(2) = \{1, 3\}$ .
- If  $v \bmod 4 = 2$ , then  $p(v) = 3$ . Thus,  $p(N_G(v)) = p(\{v-1, v+1\}) = \{2\}$ , since  $(v-1) \bmod 4 = 1$  and  $(v+1) \bmod 4 = 3$ . Thus,  $p(v-1) = 2$  and  $p(v+1) = 2$ . Therefore,  $p(N_G(v)) = \{2\}$  and  $N_R(3) = \{2\}$ .
- If  $v \bmod 4 = 3$ , then  $p(v) = 2$ . Thus,  $p(N_G(v)) = p(\{v-1, v+1\}) = \{1, 3\}$ , since  $(v-1) \bmod 4 = 2$  and  $(v+1) \bmod 4 = 0$ . Thus,  $p(v-1) = 3$  and  $p(v+1) = 1$ . Therefore,  $p(N_G(v)) = \{1, 3\}$  and  $N_R(2) = \{1, 3\}$ .

( $\Rightarrow$ ) Suppose that  $G = C_n(1)$  has an  $R_{3,1}$ -role assignment with  $N_R(1) = \{2\}$ ,  $N_R(2) = \{1, 3\}$  and  $N_R(3) = \{2\}$ . We can assume, without loss of generality, that  $p(0) = 1$ ,  $p(1) = 2$ , and  $p(2) = 3$ , since the symmetries of  $G$  allow us to start from any vertex.

We will show by induction on  $v$  that the mapping is given by:

$$p(v) = \begin{cases} 1 & \text{if } v \bmod 4 = 0, \\ 2 & \text{if } v \bmod 4 = 1 \text{ or } v \bmod 4 = 3, \\ 3 & \text{if } v \bmod 4 = 2. \end{cases}$$

Assume this holds for all  $0 \leq t \leq v$ ,  $v \geq 1$ . We show that  $p(v+1) = 1$  if  $(v+1) \bmod 4 = 0$ ,  $p(v+1) = 2$  if  $(v+1) \bmod 4 = 1$  or  $(v+1) \bmod 4 = 3$ , and  $p(v+1) = 3$ , otherwise.

- If  $(v+1) \bmod 4 = 0$ , then  $v \bmod 4 = 3$  and  $(v-1) \bmod 4 = 2$ . By the induction hypothesis,  $p(v) = 2$  and  $p(v-1) = 3$ . Since  $N_R(2) = \{1, 3\}$  and  $p(v-1) = 3$ , we conclude that  $p(v+1) = 1$ .
- If  $(v+1) \bmod 4 = 1$ , then  $v \bmod 4 = 0$  and  $(v-1) \bmod 4 = 3$ . By the induction hypothesis,  $p(v) = 1$  and  $p(v-1) = 2$ . Since  $N_R(1) = \{2\}$  and  $p(v-1) = 2$ , we conclude that  $p(v+1) = 2$ .
- If  $(v+1) \bmod 4 = 2$ , then  $v \bmod 4 = 1$  and  $(v-1) \bmod 4 = 0$ . By the induction hypothesis,  $p(v) = 2$  and  $p(v-1) = 1$ . Since  $N_R(2) = \{1, 3\}$  and  $p(v-1) = 1$ , we conclude that  $p(v+1) = 3$ .
- If  $(v+1) \bmod 4 = 3$ , then  $v \bmod 4 = 2$  and  $(v-1) \bmod 4 = 1$ . By the induction hypothesis,  $p(v) = 3$  and  $p(v-1) = 2$ . Since  $N_R(3) = \{2\}$  and  $p(v-1) = 2$ , we conclude that  $p(v+1) = 2$ .

Thus, for all  $v$ , the function  $p$  holds.

However, it is still necessary to prove that the neighborhood mapped by the function  $p$  is valid only when  $n = 4k$ . Since  $N_G(0) = \{1, n-1\}$  and  $p(0) = 1$ , we must have  $p(n-1) \in N_R(p(0)) = N_R(1) = \{2\}$ . Thus,  $p(n-1) = 2$ . This implies that  $(n-1) \bmod 4 = 1$  or  $(n-1) \bmod 4 = 3$ . Furthermore, we know that  $N_G(n-1) = \{0, n-2\}$  with  $p(n-1) = 2$ . Since  $N_R(2) = \{1, 3\}$  and we defined  $p(0) = 1$ , we conclude that  $p(n-2) = 3$ . From the definition of  $p$ ,  $p(n-2) = 3$  implies  $(n-2) \bmod 4 = 2$ . If  $(n-1) \bmod 4 = 1$ , then  $(n-2) \bmod 4 = 0$ , which contradicts  $(n-2) \bmod 4 = 2$ . If  $(n-1) \bmod 4 = 3$ , then  $(n-2) \bmod 4 = 2$ . This is consistent. Thus, the only valid solution is that  $(n-1) \bmod 4 = 3$  and  $(n-2) \bmod 4 = 2$ , which implies that  $n \bmod 4 = 0$ . Therefore, we conclude that  $\exists k \in \mathbb{N}$  such that  $n = 4k$ .  $\square$

As seen in Theorem 4.1 a  $R_{3,1}$ -role assignment for  $C_n(1)$  requires  $n$  to be a multiple of 4. The proofs for the for  $R_{3,2}$ ,  $R_{3,5}$ , and  $R_{3,7}$  are analogous, and thus omitted. Table 2 summarizes the characterizations.

Table 2: Conditions for 3-role assignments in circulant graphs  $C_n(1)$ .

| <i>R</i> -Role Assignment | Condition on $n$ |
|---------------------------|------------------|
| $R_{3,1}$                 | $n = 4k$         |
| $R_{3,2}$                 | $n = 5k$         |
| $R_{3,5}$                 | $n = 6k$         |
| $R_{3,7}$                 | $n = 3k$         |

## 5. Circulant Graphs with $r$ -Role Assignment for $r \geq 3$

In this section, we generalize the obtained results in Section 4 to  $r$ -role assignments for  $r \geq 3$  in circulant graphs of type  $C_n(1)$ . Figure 3 presents all possible connected role graphs. Each subgraph is denoted as  $R_{r,i}$ , where  $r$  represents the number of roles and  $i$  the index of the role graph.

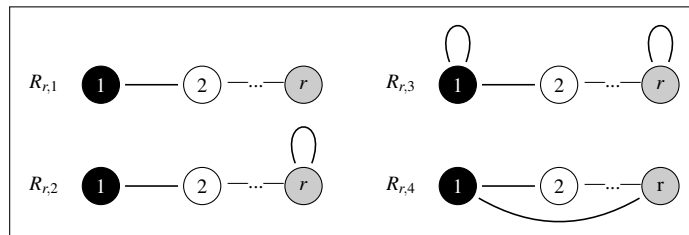


Figure 3: Possible connected role graphs arising from an  $r$ -role assignment,  $r \geq 3$ , for  $C_n(1)$ .

We proceed by showing in Theorem 5.1 a necessary and sufficient condition for  $C_n(1)$  to admit an  $R_{r,1}$ -role assignment.

**Theorem 5.1.** *The graph  $G = C_n(1)$  admits an  $R_{r,1}$ -role assignment with  $r \geq 3$  if and only if  $n = (2r-2)k$ , for some  $k \in \mathbb{N}$ .*

*Proof.* ( $\Leftarrow$ ) Let  $n = (2r-2)k$ . We can define the function  $p : V(G) \rightarrow \{1, 2, \dots, r\}$  as follows:

$$p(v) = \begin{cases} (v \bmod (2r-2)) + 1, & \text{if } v \bmod (2r-2) < r, \\ 2r - ((v \bmod (2r-2)) + 1), & \text{otherwise.} \end{cases}$$

We show that  $p$  is an  $R_{r,1}$ -role assignment. This requires verifying the six cases described in Table 3.

Table 3: Cases considered to show that  $p$  is an  $R_{r,1}$ -role assignment.

| Case | $v \bmod (2r-2)$   | $p(v) = i$      | $p(v-1)$ | $p(v+1)$ |
|------|--------------------|-----------------|----------|----------|
| 1    | 0                  | 1               | 2        | 2        |
| 2    | $1, \dots, r-2$    | $2, \dots, r-1$ | $i-1$    | $i+1$    |
| 3    | $r-1$              | $r$             | $r-1$    | $r-1$    |
| 4    | $r$                | $r-1$           | $r$      | $r-2$    |
| 5    | $r+1, \dots, 2r-4$ | $r-2, \dots, 3$ | $i+1$    | $i-1$    |
| 6    | $2r-3$             | 2               | 3        | 1        |

**Case 1:** If  $v \bmod (2r-2) = 0$ , then  $p(v) = 1$ . Recall that  $N_R(1) = \{2\}$ .

For  $v-1$ , since  $(v-1) \bmod (2r-2) = 2r-3 \geq r$ , we have:  $p(v-1) = 2r - (2r-3+1) = 2$ .

For  $v+1$ , since  $(v+1) \bmod (2r-2) = 1 < r$ , we have:  $p(v+1) = 1 + 1 = 2$ .

Therefore,  $p(N_G(v)) = \{2\} = N_R(p(1))$ .

**Case 2:** If  $1 \leq v \bmod (2r-2) \leq r-2$ , then  $2 \leq p(v) \leq r-1$ . Thus,  $0 \leq (v-1) \bmod (2r-2) \leq r-3$  and  $2 \leq (v+1) \bmod (2r-2) \leq r-1$ . Note that for  $v \bmod (2r-2) \leq r-2$ ,  $(v+1) \bmod (2r-2) \leq r-1$ .

Since  $v-1$  and  $v+1$  will be less than or equal to  $r-1$  in this case, we have:

$$\begin{cases} p(v-1) = ((v-1) \bmod (2r-2)) + 1 = (v \bmod (2r-2)) + 1 - 1 = p(v) - 1 \\ p(v+1) = ((v+1) \bmod (2r-2)) + 1 = (v \bmod (2r-2)) + 1 + 1 = p(v) + 1 \end{cases}$$

Thus,  $p(N_G(v)) = \{p(v)-1, p(v)+1\} = N_R(p(v))$ .

**Case 3:** If  $v \bmod (2r-2) = r-1$ , then  $p(v) = r$ . We have  $N_R(r) = \{r-1\}$ .

For  $v-1$ , since  $(v-1) \bmod (2r-2) = r-2 < r$ , we have:  $p(v-1) = (r-2) + 1 = r-1$ .

For  $v+1$ , since  $(v+1) \bmod (2r-2) = r \geq r$ , we have:  $p(v+1) = 2r - (r+1) = r-1$ .

Thus,  $p(N_G(v)) = \{r-1\} = N_R(p(r))$ .

**Case 4:** If  $v \bmod (2r-2) = r$ , then  $p(v) = r-1$ . Given that  $N_R(r-1) = \{r, r-2\}$ ,

For  $v-1$ , since  $(v-1) \bmod (2r-2) = r-1 < r$ , we have:  $p(v-1) = (r-1) + 1 = r$ .

For  $v+1$  and  $r > 3$ , when  $(v+1) \bmod (2r-2) = r+1 \geq r$ , we have:  $p(v+1) = 2r - (r+2) = r-2$ .

When  $r = 3$ , we have  $(v+1) \bmod (2r-2) < r$ , since  $(v+1) = r+1 = 4$  and  $2r-2 = 4$ . In this case,  $p(v+1) = 1$ , and  $r-2 = 3-2 = 1$ .

Therefore  $p(N_G(v)) = \{r, r-2\} = N_R(p(v))$ .

**Case 5:** If  $r+1 \leq v \bmod (2r-2) \leq 2r-4$ , then  $3 \leq p(v) \leq r-2$ . Note that  $p(v) = 2r - ((v \bmod (2r-2)) + 1)$ . Since  $r+1 \leq v \bmod (2r-2) \leq 2r-4$ ,  $r+2 \leq (v \bmod (2r-2)) + 1 \leq 2r-3$ . Thus  $2r - (2r-3) \leq p(v) \leq 2r - (r+2)$ , which means  $3 \leq p(v) \leq r-2$ .

Thus,  $r \leq (v-1) \bmod (2r-2) \leq 2r-5$  and  $r+2 \leq (v+1) \bmod (2r-2) \leq 2r-3$ . Since  $v-1$  and  $v+1$  will be greater than or equal to  $r$ , we have:

$$\begin{cases} p(v-1) = 2r - ((v-1) \bmod (2r-2) + 1) = 2r - (v \bmod (2r-2) + 1 - 1 + 1) \\ \quad \quad \quad = (2r - (v \bmod (2r-2) + 1)) + 1 = p(v) + 1 \\ p(v+1) = 2r - ((v+1) \bmod (2r-2) + 1) = 2r - (v \bmod (2r-2) + 1 + 1) \\ \quad \quad \quad = (2r - (v \bmod (2r-2) + 1)) - 1 = p(v) - 1 \end{cases}$$

Thus,  $p(N_G(v)) = \{p(v)-1, p(v)+1\} = N_R(p(v))$ .

**Case 6:** If  $v \bmod (2r-2) = 2r-3$ , then  $p(v) = 2$ . Given that  $N_R(2) = \{1, 3\}$ ,

For  $v-1$ , since  $v-1 \bmod (2r-2) = 2r-4 \geq r$ , we have:  $p(v-1) = 2r - (2r-4+1) = 3$ .

For  $v+1$ , since  $(v+1) \bmod (2r-2) = 0 < r$ , we have:  $p(v+1) = 1$ .

Therefore,  $p(N_G(v)) = \{1, 3\} = N_R(p(2))$ .

( $\Rightarrow$ ) Suppose there exists a function  $p : V(C_n(1)) \rightarrow \{1, 2, \dots, r\}$  that is an  $R_{r,1}$ -role assignment. By symmetry, we can assume that  $p(0) = 1$  and  $p(1) = 2$ . We make use of an auxiliary function  $f$  for a better organization of the text. Our aim is to show that  $p = f$  where:

$$f(v) = \begin{cases} (v \bmod (2r-2)) + 1, & \text{if } v \bmod (2r-2) < r, \\ 2r - ((v \bmod (2r-2)) + 1), & \text{otherwise.} \end{cases}$$

We proceed by induction. Assume that  $p(v-1) = f(v-1)$  and  $p(v) = f(v)$  for some  $1 \leq v \leq n-2$ .

From the sufficiency proof, we know that  $f$  satisfies  $f(N_G(v)) = N_R(f(v))$ . Furthermore, since  $p$  is a role assignment, we have  $p(N_G(v)) = N_R(p(v))$ . Consequently,  $p(N_G(v)) = f(N_G(v))$ .

Given that  $N_G(v) = \{v-1, v+1\}$ , it follows that  $\{p(v-1), p(v+1)\} = \{f(v-1), f(v+1)\}$ .

Note that these sets can have one or two elements. In both cases, since we have  $p(v-1) = f(v-1)$ , we conclude that  $p(v+1) = f(v+1)$  for all  $0 \leq v \leq n-1$ .

Therefore,  $p(v+1) = f(v+1)$ .

Note that the only vertices that may not satisfy  $f(N_G(v)) = N_R(v)$  are  $v = 0$  or  $v = n-1$ . Since  $n-1 \in N_G(0)$  and  $p(0) = 1$ , we have  $f(n-1) = 2$ . Thus,  $f(n-2) = p(n-2) = 3$ . We observe that this implies  $(n-1) \bmod (2r-2) \geq r$ .

$$\begin{aligned} 2 &= f(n-1) = 2r - (((n-1) \bmod (2r-2)) + 1), \\ &\Rightarrow (n-1) \bmod (2r-2) = 2r-1-2 = 2r-3, \\ &\Rightarrow n-1 = 2r-3 + (2r-2)k', \quad \text{where } k' \in \mathbb{N}_0, \\ &\Rightarrow n = 2r-2 + (2r-2)k' = (1+k')(2r-2). \end{aligned}$$

Letting  $k = 1 + k'$ , we have  $n = k(2r-2)$  for some  $k \in \mathbb{N}$ . □

Theorem 5.1 states that a  $R_{r,1}$ -role assignment for  $C_n(1)$  requires  $n = (2r-2)k$ , for some  $k \in \mathbb{N}$ . The proofs for the conditions for  $R_{r,2}$ ,  $R_{r,3}$ , and  $R_{r,4}$  are analogous and are therefore omitted. To put everything together, Table 4 summarizes the results.

Table 4: Conditions for  $r$ -role assignments,  $r \geq 3$ , in circulant graphs  $C_n(1)$ .

| <b><math>R</math>-Role Assignment</b> | <b>Condition on <math>n</math></b> |
|---------------------------------------|------------------------------------|
| $R_{r,1}$                             | $n = (2r - 2)k$                    |
| $R_{r,2}$                             | $n = (2r - 1)k$                    |
| $R_{r,3}$                             | $n = 2rk$                          |
| $R_{r,4}$                             | $n = rk$                           |

## 6. Concluding Remarks

In this work, we have studied circulant graphs and their properties with an emphasis on  $r$ -role assignments. The presented analysis has shown that role assignments in circulant graphs  $C_n(1)$  are directly related to the number of vertices  $n$  and  $r$ . For future work, we suggest to extend the study to circulant graphs with multiple parameters  $d_\ell$ , evaluating how the presence of additional edges affects the conditions for a role assignment.

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