

Independent identifying codes: complexity and closed formulas for some graph classes

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ABSTRACT

An identifying code of a graph is a subset of its vertices such that every vertex of the graph is uniquely identified by the set of its neighbors within the code. An independent identifying code (an IIC set) is an identifying code that is also independent. This paper investigates the problem of determining an independent identifying code in a given nontrivial simple undirected connected graph. We prove that the problems of determining the existence of IIC sets in planar bipartite graphs and in graphs with maximum degree four are $\mathcal{NP}$ -complete. Furthermore, we study IIC sets in certain infinite families of snarks – namely, flower snarks, generalized Blanuša snarks, and Goldberg snarks. For the first two families, we provide closed formulas for the size of an IIC set; for the third, we prove that no such code exists.

KEYWORDS. Snark, Complexity, Independent, Identifying code, Flower, Blanuša, Goldberg.

PAPER TOPICS. Independent identifying codes, Planar bipartite graphs, {Flower, generalized Blanuša, Goldberg} Snarks, Complexity.

1. Introduction

Consider the situation where a graph G models a facility or a multiprocessor network with limited-range detection devices (sensing for example, movement, heat or size) placed at chosen nodes of G . The purpose of these devices is to detect and precisely identify the location of a *failure* (e.g., undesirable encroachment by a thief, saboteur or vandal, fire or faulty processor) that may suddenly occur at any node. Since it is costly to install and maintain such detection devices, it is logical to determine the location at the nodes of the minimum number of devices at the nodes that, together, can accurately identify such a failure at any location. A device at any location can detect whether the fault occurs at its own location or in the open neighborhood of its own location, but which location in the neighborhood cannot be accurately determined. It is assumed that each detector cannot be located in the range of any other because of the possible interference of the signals from the two detectors, i.e. the set of nodes chosen as detector locations is independent. Such a situation can potentially be analyzed via *independent identifying codes* [Slater and Sewell, 2018].

Throughout this paper let $G = (V(G), E(G))$ be a nontrivial simple undirected connected graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$ is defined to be the set of neighbors of v in G and is denoted by $N(v)$. Thus $N(v) = \{u \mid uv \in E(G)\}$. The *closed neighborhood* of v is denoted by $N[v]$, where $N[v] = N(v) \cup \{v\}$. The *vertex subset neighborhood* is denoted by $N[A] = \bigcup_{u \in A} N[u]$, $A \subseteq V(G)$. For $u, v \in V(G)$, u and v are *twins* if $N[u] = N[v]$ and *false twins* if $N(u) = N(v)$. A graph with n vertices that is a path, respectively a cycle, is termed an n -path, respectively, an n -cycle, and is denoted by P_n ,

respectively by C_n . For the sake of notation simplification, let $[k]$ denote the set $\{1, 2, \dots, k\}$ for any positive integer k .

A vertex subset $S \subset V(G)$ is termed *dominating* if every vertex of G not in S is adjacent to at least one member of S . The cardinality of the smallest dominating set is denoted by $\gamma(G)$. To generalize dominating sets, S is termed *2-dominating* if every vertex of G not in S is adjacent to at least two members of S . The set S is termed *independent* if no two vertices in S are adjacent in G . S is termed *independent dominating (ID)* if it is independent and dominates $V(G)$. The cardinality of the smallest independent dominating set is denoted by $i(G)$. The set S is *separating* if for each pair $u, v \in V(G)$ ($u \neq v$), $N[u] \cap S \neq N[v] \cap S$. In this case, we say that set S *separates* all vertices of G . Also, S is termed an *identifying code (IC)* if it is dominating and separating. A graph G does not have an IC if it has twin vertices. The cardinality of the smallest identifying code is denoted by $IC(G)$. S is termed an *independent identifying code (IIC set)* if it is both independent and an identifying code. The cardinality of the smallest IIC set of G is denoted by $IIC(G)$. Not all graphs have an IIC set, such as the graphs C_3 and C_4 and those shown in Figures 1(a) and 1(b). However, the graph in Figure 1(c) has an IIC set. The problem of determining whether a given arbitrary graph contains an IIC set is an $\mathcal{NP}$ -complete problem [Slater and Sewell, 2018].

Identifying codes were first introduced by Karpovsky et al. [1998], where the authors related the problem to the diagnosis of faults in multiprocessor systems. Identifying codes have been widely studied, and a detailed list of references on this subject is maintained by Jean [2023]. From a computational point of view, finding identifying codes of minimum cardinality has been proved to be $\mathcal{NP}$ -hard [Charon et al., 2003]. Therefore, it is natural that many researchers have restricted their study of identifying codes in some specific classes of graphs such as trees [Bertrand et al., 2005], cycles [Gravier et al., 2006], and grids [Junnala and Laihonon, 2012; Dantas et al., 2017], for instance.

A *cubic (3-regular)* graph is a graph where every vertex is of degree 3. For an arbitrary graph G , determining $i(G)$ is $\mathcal{NP}$ -hard even for bipartite cubic graphs [Liu et al., 2015]. A *snark* is a cubic graph that is both bridgeless and non-3-edge-colorable [Brinkmann et al., 2013; Cavicchioli et al., 1998]. Snarks play a crucial role in graph theory, particularly in the study of the four-color theorem, cycle double covers, and various conjectures related to edge-coloring and flow theory. The non-3-edge-colorability of snarks makes them essential counterexamples in extremal graph theory, as they challenge the boundaries of known coloring results. The smallest known snark is the Petersen graph. Pereira and Campos [2024] established bounds on the domination numbers and independent domination numbers of both the families of generalized Blanuša and Loupekin snarks.

In the present work, we investigate the problem of determining IIC sets in particular classes of simple connected graphs, with a particular focus on its computational complexity. We present some preliminary results in Section 2. In Section 3, we prove that the problem remains

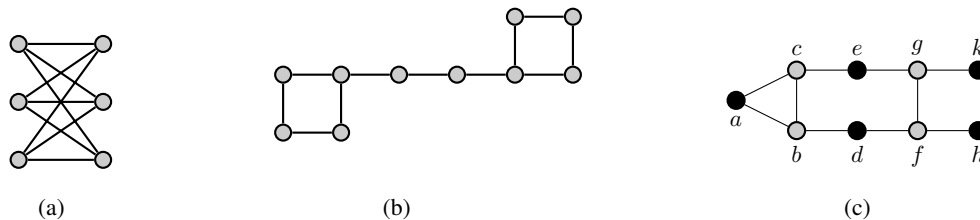


Figure 1: The graphs in (a) and (b) do not have an IIC set, but the graph in (c) has the IIC set $S = \{a, d, e, k, h\}$.

$\mathcal{NP}$ -complete even when restricted to planar bipartite graphs or graphs of bounded degree four. Section 4 explores three infinite families of snarks, where we determine the exact size of the IIC set for two of them, while for the third, we prove that it does not have an IIC set. We round out the paper with a conclusion in Section 5.

2. Preliminary results

In this section, we present some preliminary results on IIC sets. First, we establish some conditions for S being an IIC set of a graph G .

Lemma 1 (Slater and Sewell [2018]). *If S is an IIC set for a graph G and $v \in V(G)$ has $\deg(v) = 1$, then $v \in S$.*

Observation 2. *If S is an IIC set for a graph G and $u, v \in V(G)$ are false twins, then $u, v \in S$.*

By Theorem 3, every IIC set S of a graph G is also a 2-dominating set, if it exists.

Theorem 3. *If a simple graph G has an IIC set S and there exists a vertex $u \in V(G) \setminus S$, then $|N[u] \cap S| \geq 2$.*

Proof. Suppose that S is an IIC set of G and there exists a vertex $u \in V(G) \setminus S$. Then, as S dominates G , there exists a vertex v say, in S such that $v \in N[u]$. Therefore, by contradiction, if v is the only vertex in S that dominates u , then as S is independent, u and v would not be separated by S . Hence, there must be an additional vertex (which is in $S \setminus \{v\}$) that dominates u , and the result follows. $\square$

The following theorem follows directly from Lemma 1 and Theorem 3.

Theorem 4. *For path P_n with $n \geq 5$ and odd, $IIC(P_n) = \lceil \frac{n}{2} \rceil$. Paths P_1 , P_3 , and P_n with $n \geq 2$ and even, do not have IIC sets. For $n \geq 6$ and even, $IIC(C_n) = \frac{n}{2}$. Cycles C_3 , C_4 , and C_n with $n \geq 5$ and odd, do not have IIC sets.*

3. Complexity results

The IIC problem of establishing whether or not a given arbitrary graph has an independent identifying code can be formally stated as follows:

Instance – Simple graph $G = (V, E)$.

Question – Does G admit an IIC set $S \subseteq V(G)$?

This problem was first introduced by Slater and Sewell [2018], who settled the general question of complexity, as stated in the theorem below.

Theorem 5 (Slater and Sewell [2018]). *Deciding, for a given graph G , if G has an IIC set is an $\mathcal{NP}$ -complete problem.*

The following theorem shows that the IIC problem remains $\mathcal{NP}$ -complete for the special case of planar bipartite graphs by providing a reduction from the PLANAR 3-SAT decision problem.

Theorem 6. $\mathcal{BP}IIC$: *Deciding, for a given planar bipartite graph G , if G has an IIC set is an $\mathcal{NP}$ -complete problem.*

Proof. Membership in $\mathcal{NP}$ was proved by [Slater and Sewell, 2018]. We now show that the PLANAR 3-SAT problem is reducible in polynomial time to **BPTIC**. Let $U = \{u_1, u_2, \dots, u_n\}$ and $C = \{c_1, c_2, \dots, c_m\}$ be an instance ϕ of the PLANAR 3-SAT. We construct a graph G as follows. For each variable $u_i \in U$, add a copy of graph H_i shown in Figure 2(a). For each clause $c_j \in C$, add a copy of graph Z_j shown in Figure 2(b). Then, each vertex α_j corresponding to a clause c_j is made adjacent to the three literal vertices that clause c_j contains. The resulting graph G has order $|V(G)| = 16n + 2m$ and size $|E(G)| = 20n + m$, and can be constructed in polynomial time.

Clearly, G is planar because ϕ is an instance of PLANAR 3-SAT. Furthermore, G is bipartite as H_i and Z_j are bipartite graphs, and u_i and $\bar{u}_i$ are in the same part of $V(G)$, say A , and α_j is in the other part, say B . We prove that (i) G has an IIC set if and only if (ii) ϕ is satisfiable.

To prove that (i) is necessary for (ii), assume that ϕ has a satisfying truth assignment. Let $D_1 = \{u_i, b_i, c_i, k_i, \ell_i, o_i \mid u_i \text{ is True}\}$ and $D_2 = \{\bar{u}_i, a_i, d_i, e_i, q_i, p_i \mid u_i \text{ is False}\}$. Let $S = \{s_i, g_i\} \cup D_1 \cup D_2 \cup \{\beta_j \mid 1 \leq j \leq m\}$. Then it is straightforward to see that S is an IIC set of G .

To prove that (ii) is necessary for (i), assume that S say, is an IIC set of G . Observe that $\{s_i, g_i\} \subseteq S$ for all $i \in [n]$ and $\{\beta_j\} \subseteq S$ for all $j \in [m]$. Moreover, S does not contain both u_i and $\bar{u}_i$, otherwise $e_i, k_i \notin S$, which is impossible since in this case h_i and g_i are not separated by S . So S is not an IIC set of G . If $\{u_i, \bar{u}_i\} \cap S = \emptyset$, then h_i and s_i are not separated by S , and, again S is not an IIC set of G . So, we can conclude that $|S \cap \{u_i, \bar{u}_i\}| = 1$. By independence, $\alpha_j \notin S$. Since $N[\beta_j] \cap S \neq N[\alpha_j] \cap S$, α_j is adjacent to one of the literal vertices v say, corresponding to a literal contained in clause c_j with $v \in C$. Thus, defining u_i as True if and only if $u_i \in S$ produces a satisfying truth assignment for ϕ . $\square$

In order to prove Theorem 5, Slater and Sewell [2018] presented a similar reduction from 3-SAT, using the gadget G_i presented in Figure 2(c) for the variables and the gadget for the clause in Figure 2(b). By adapting the reduction from Slater and Sewell [2018], using the $\mathcal{NP}$ -complete problem P3-SAT $_{\bar{3}}$ from Berman et al. [2003], described below, we can prove that the IIC problem is $\mathcal{NP}$ -complete, even for graphs of maximum degree four, as we state in Corollary 7.

P3-SAT $_{\bar{3}}$ – Planar 3-SAT with at most three occurrences per variable.

Instance – A set U of variables and a collection C of clauses over U , where $|U| = n$ and $|C| = m$, such that: (i) each clause $c \in C$ satisfies $|c| = 2$ or $|c| = 3$; (ii) each variable has two or three occurrences and each negative literal occurs once in C ; (iii) the bipartite undirected graph $G = (V, E)$ is planar and connected, where $V = U \cup C$ and E contains edge uc if and only if either u or $\bar{u}$ belongs to clause c .

Question – Is there a satisfying truth assignment for U satisfying each clause of C ?

Corollary 7. Deciding, for a given graph G of maximum degree four, if G has an IIC set is an $\mathcal{NP}$ -complete problem.

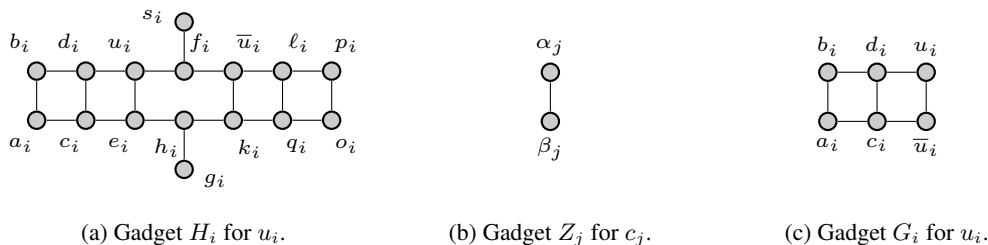


Figure 2: The gadgets used in the reductions to the **IIC** problem.

4. Results for snarks

In this section, we study the IIC problem in three infinite families of snarks: flower snarks, generalized Blanuša snarks and Goldberg snarks. While flower and generalized Blanuša snarks admit IIC sets, we prove that all Goldberg snarks do not.

4.1. Flower snarks

Let J_n be a flower snark graph constructed as follows. Add n copies of the graph $K_{1,3}$, represented by: $S_0, S_1, \dots, S_{n-1}$, where c_i is the central vertex of the i^{th} copy, and a_i, b_i , and d_i are its pendant vertices; add edges to form the cycle $\langle d_1, d_2, \dots, d_n \rangle$; and additional edges to create the cycle $\langle a_1, \dots, a_n, b_1, \dots, b_n \rangle$. For simplicity, we shall also denote S_i as the set of vertices of S_i .

Lemma 8. *Let S be an IIC set of J_n . For every S_i , $1 \leq i \leq n$, $|S_i \cap S| \geq 1$.*

Proof. For each S_i to dominate vertex c_i it is necessary that either $c_i \in S$ or $|N(c_i) \cap S| \geq 1$. $\square$

Lemma 9. *Let S be an IIC set of J_n . If $|S_i \cap S| = 3$, then $|S_{i-1} \cap S| = |S_{i+1} \cap S| = 1$.*

Proof. By the independence property, $S_u \cap S = \{a_u, b_u, d_u\}$, implying that $a_{u-1}, b_{u-1}, d_{u-1}, a_{u+1}, b_{u+1}, d_{u+1} \notin S$. Thus, $c_{u-1}, c_{u+1} \in S$ for domination. The set $\{c_{u-1}, a_u, b_u, d_u, c_{u+1}\}$ remains independent and also dominates and separates S_{u-1}, S_u and S_{u+1} , and is a subset of S . $\square$

Corollary 10. *If S is an IIC set of J_n , where n is a multiple of 3, then $|S| \leq \frac{5n}{3}$.*

Proof. Let S' be the set where, for $i \in [n]$, we have $S_i \cap S = \{a_i, b_i, d_i\}$ if $i \bmod 3 = 1$, or $S_i \cap S = \{c_i\}$, otherwise. (S' corresponds to the choice expressed in Lemma 9.) Thus, S' is a valid IIC set for J_n , and $|S'| = \frac{5n}{3}$, since dominating and separating every three adjacent blocks S_i requires five vertices. $\square$

Lemma 11. *Let S be an IIC set of J_n . If $|S_i \cap S| = 1$ then $c_i \in S$ and $a_i, b_i, d_i \notin S$.*

Proof. Suppose $u \in S$, where $u \in \{a_i, b_i, d_i\}$. Then, in order to ensure that $N[u] \cap S \neq N[c_i] \cap S$, it is necessary that $v \in S$, where $v \in \{a_i, b_i, d_i\} \setminus \{u\}$. $\square$

Lemma 12. *Let S be an IIC set of J_n . If $|S_i \cap S| = |S_{i+1} \cap S| = 1$ then $|S_{i-1} \cap S| = |S_{i+2} \cap S| = 3$.*

Proof. Since $|S_i \cap S| = |S_{i+1} \cap S| = 1$, it follows from Lemma 11 that $c_i, c_{i+1} \in S$. Therefore, to ensure that $N[a_i] \cap S, N[b_i] \cap S$, and $N[d_i] \cap S$ all differ from $N[c_i] \cap S$, it is necessary that a_{i-1}, b_{i-1} , and c_{i-1} all belong to S . Similarly, to ensure that $N[a_{i+1}] \cap S, N[b_{i+1}] \cap S$, and $N[d_{i+1}] \cap S$ differ from $N[c_{i+1}] \cap S$, it is necessary that a_{i+2}, b_{i+2} , and c_{i+2} are all in S . Hence, $|S_{i-1} \cap S| = |S_{i+2} \cap S| = 3$. $\square$

Lemma 13. *If S is an IIC set of J_n , then $|S_i \cap S| \neq 2$.*

Proof. By contradiction. Suppose there exists some S_i such that $|S_i \cap S| = 2$. Due to the independence property, $S_i \cap S$ can be either $\{a_i, b_i\}$, $\{a_i, d_i\}$, or $\{b_i, d_i\}$. Let us assume, w.l.o.g., $S_i \cap S = \{a_i, b_i\}$. Observe that $a_j, b_j \notin S$ for $j \in \{i-1, i+1\}$, since they are adjacent to a_i and b_i , respectively. Moreover, $d_j \notin S$, as otherwise, this would force $S_j \cap S = \{d_j\}$, which is impossible due to Lemma 11. Therefore, only c_j can be added to S . However, the vertices in $\{c_{i-1}, a_i, b_i, d_i, c_{i+1}\} \subseteq S$ do not dominate d_i , and thus, S cannot be a valid IIC. $\square$

Theorem 14. $IIC(J_n) = n + 2 \lfloor \frac{n+2}{3} \rfloor$.

Proof. Let S be an IIC set of J_n . We construct S such that, for every $0 \leq i \leq n-1$,

$$S_i \cap S = \begin{cases} \{a_i, b_i, d_i\}, & \text{if } i \bmod 3 = 1; \\ \{c_i\}, & \text{otherwise.} \end{cases}$$

Note that, when n is a multiple of 3, this construction yields the same result as that presented in Corollary 10. Moreover, it produces an optimal configuration by always attempting to maximize the number of blocks S_i for which $|S_i \cap S| = 1$. According to Lemma 12, no more than two such consecutive blocks are allowed. Thus, every third block must contain three vertices in S , which is optimal by Lemma 13. This pattern can then be repeated in the graph. Figure 3 illustrates this construction for the flower snark J_{10} . $\square$

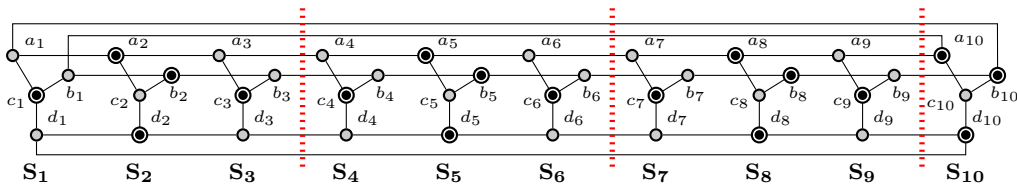


Figure 3: The minimum IIC set of J_{10} , $IIC(J_{10}) = 18$.

4.2. Generalized Blanuša Snarks

The *Generalized Blanuša snarks* consist of two families of snarks, denoted by $\mathcal{B}^1 = \{B_1^1, B_2^1, B_3^1, \dots\}$ and $\mathcal{B}^2 = \{B_1^2, B_2^2, B_3^2, \dots\}$. Each graph B_n^1 , $n \in \mathbb{N}^*$, is defined as the union of the base block graph B_0^1 (Figure 4(a)) and n copies of the block graph B (Figure 4(c)), with the edges $\{(c_{i-1}, f_i), (h_{i-1}, a_i) \mid i \in \{1, \dots, n\}\} \cup \{(t_0, h_n), (w_0, c_n)\}$ added to maintain connectivity. The graph B_n^2 is constructed similarly, but using the base block B_0^2 (Figure 4(b)) and the edges $\{(c_{i-1}, f_i), (h_{i-1}, a_i) \mid i \in \{1, \dots, n\}\} \cup \{(a_0, h_n), (f_0, c_n)\}$ to connectivity [Pereira and Campos, 2024]. The total number of vertices in both B_n^1 and B_n^2 is $|V(B_n^1)| = |V(B_n^2)| = 8n + 10$. A vertex within a block is termed *internal* if it has no neighbors in other blocks. Let $I(B_0^1) = \{a_0, b_0, d_0, e_0, f_0, g_0\}$, $I(B_0^2) = \{b_0, d_0, e_0, g_0, t_0, w_0\}$ and $I(B_i) = \{b_i, d_i, e_i, g_i\}$ be the sets of internal vertices of blocks graphs B_0^1 , B_0^2 and B_i , respectively (see Figure 4).

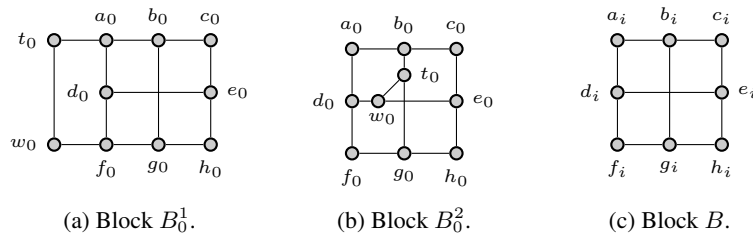


Figure 4: The building blocks of the generalized Blanuša snarks B^1 and B^2 .

Theorem 15. $IIC(B_n^1) = 4n + 4 - \lfloor \frac{2n}{3} \rfloor$ for $n > 1$.

Proof. The vertices in $I(B)$ can be separated and dominated only by vertices within the same block B . Based on this, only the configurations illustrated in Figure 5, labeled from A to E, are valid for separating and dominating B blocks. The dotted edges represent connections to other blocks, and the red vertices indicate vertices from other blocks that must be included in the IIC set. Note

that configurations *A* through *D* require only three vertices within the block, while configuration *E* requires four. Similarly, the only valid configurations for block B_0^1 , labeled **X**, **Y**, **Z**, **T** and **W** (in bold), are shown in Figure 6, where they all require four vertices within the block.

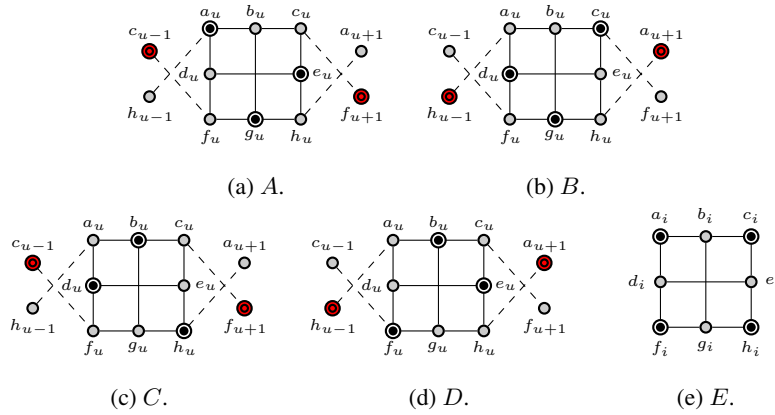


Figure 5: The possible choices for block *B* of the Blanuša Snark B^1 .

Consider two consecutive blocks B_i and B_{i+1} in a B_n^1 structure, with $n \geq 2$, and let S be one of their *IIC* sets. Note that configuration *E* cannot be used for B_i together with configuration *A* for B_{i+1} , since vertices h_i and a_i would both belong to S , leading to a contradiction because h_i and a_i are adjacent. A similar issue arises when using configuration *D* for B_i and configuration *C* for B_{i+1} , as configuration *C* requires vertex a_{i+1} to be in S . Thus, we can extend this type of analysis to each possible pair of configurations for blocks of types *B* and B_0^1 , identifying which configurations can coexist without conflict when transitioning from B_i to B_{i+1} . The results of this analysis are summarized in Table 1.

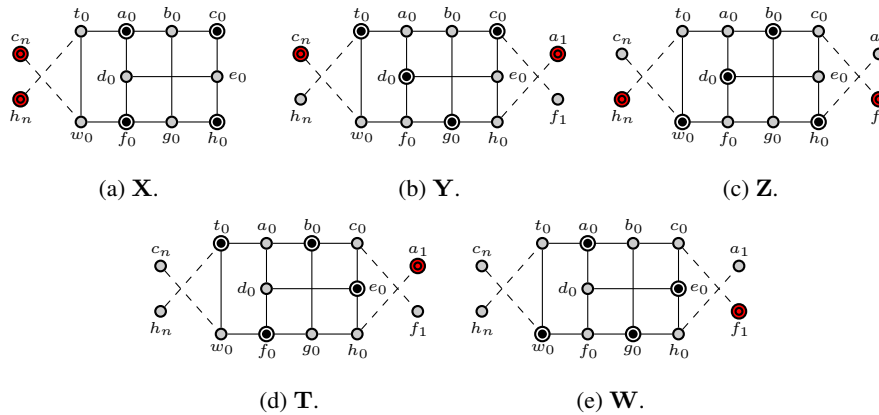


Figure 6: The possible choices for block *B* of the Blanuša Snark B_0^1 .

The information in Table 1 can be interpreted as the adjacency list of a directed graph, where the column “Config.” represents each individual vertex, and the column “Next config.” lists the vertices reachable from it. In this way, we can see that any walk of length $m > 1$, starting and ending at a vertex $u \in H = \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}, \mathbf{W}\}$ and not passing through any other vertex in $H \setminus \{u\}$, represents an *IIC* set of the graph B_m^1 , i.e. a valid sequence of configurations for consecutive blocks beginning and ending at block B_0^1 . Each time a vertex from the set $\{A, B, C, D\}$ appears

in the walk, three vertices are added to the *IIC* set of B_m^1 . Similarly, each visit to configuration E adds four vertices to the *IIC* set. Additionally, there are four more vertices corresponding to the separation and domination of the block B_0^1 itself. In other words, if P is a walk that starts and ends at $u \in H$ and does not visit the vertices in $H \setminus \{u\}$, the i^{th} vertex in P represents the configuration of the i^{th} B block in the graph $B_{|P|}^1$, and u represents the configuration of the block B_0^1 in $B_{|P|}^1$.

Table 1: The possible configurations of adjacent states for blocks of B_n^1 and B_n^2 .

Blocks of B_n^1				Blocks of B_n^2			
Config.	Next config.	Config.	Next config.	Config.	Nex config.	Config.	Next config.
A	E, W	X	B, C	A	E, Z	X	E
B	A, Y, T	Y	A	B	A, T	Y	B, C
C	D, Z, W	Z	D	C	D, W	Z	E
D	E, T	T	E, A	D	E, Z	T	A
E	B, C, X	W	E, D	E	B, C, X, Y	W	D

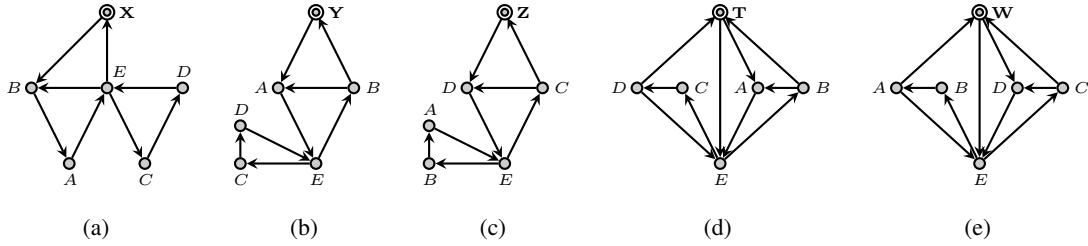


Figure 7: The transition graphs of the configurations in B_n^1 .

Figures 7(a) to 7(e) show each of the graphs from the first part of Table 1, focusing individually on the configurations: X , Y , Z , T and W for the block B_0^1 , respectively. Let S be an *IIC* set for B_n^1 . We now analyze the walks in the graphs of Figure 7 that define *IIC* sets for graphs B_n^1 .

- The walks when $n \bmod 3 = 0$, $n \geq 3$:

$$\begin{aligned}
 &XBAE(BAE \mid CDE)^*X, & TE(CDE \mid BAE)^*CDT, \\
 &YAE(CDE \mid BAE)^*BY, & WDE(CDE \mid BAE)^*CW, \\
 &ZDE(BAE \mid CDE)^*CZ, & WE(BAE \mid CDE)^*BAW, \\
 &TAE(CDE \mid BAE)^*BT.
 \end{aligned} \tag{0}$$

All the walks include the configuration assigned to block B_0^1 , which contributes four vertices to the set S . Additionally, for every consecutive sequence of three B blocks, each walk contains one configuration of type E (using four vertices) and two configurations of other types (each using three vertices), resulting in a total of $|S| = \frac{10n}{3} + 4$.

- The walks when $n \bmod 3 = 1$, $n > 1$:

$$TAE(BAE \mid CDE)^*CDT, \quad WDE(CDE \mid BAE)^*BAW. \tag{1}$$

The repetition of the sequence inside each parenthesis in (1) accounts for 10 vertices, and an additional seven vertices are required for the head, and for the segments before and after the

loop. Thus, we have $|S| = \lfloor \frac{10n}{3} \rfloor + 8$. Note that there is no valid walk involving only two vertices, which means that no *IIC* set exists for the graph B_1^1 .

- The walks when $n \bmod 3 = 2$:

$$TE(BAE \mid CDE)^*BT, \quad WE(CDE \mid BAE)^*CW. \quad (2)$$

Similarly to the previous case, each repetition of the sequence inside each parenthesis in (2) adds 10 vertices to the *IIC* set. In addition, four vertices are required for the configuration T of block B_0^1 , four for the configuration E , and three more for the configuration following the loop. This results in a total of $|S| = \lfloor \frac{10n}{3} \rfloor + 11$.

Observe that the symbol “ $\mid$ ” separates alternative sequences that can be chosen at that point, and $(\dots)^*$ means that the grouped sequence may repeat any number of times, including zero times (i.e. never). From this, it follows that the minimum *IIC* sets of B_n^1 correspond to the patterns in (0), (1) or (2), when $n \bmod 3 = 0, 1$ or 2 , respectively. Therefore, we can write:

$$IIC(B_n^1) = \begin{cases} \frac{10n}{3} + 4, & \text{if } n \bmod 3 = 0; \\ \lfloor \frac{10n}{3} \rfloor + 8, & \text{if } n \bmod 3 = 1; \\ \lfloor \frac{10n}{3} \rfloor + 11, & \text{if } n \bmod 3 = 2. \end{cases} \quad \equiv \quad IIC(B_n^1) = 4n + 4 - \left\lfloor \frac{2n}{3} \right\rfloor. \quad (3)$$

Figure 8 shows an example of the graph B_8^1 , using the *IIC* set given by the walk $TEBAEBAEBT$, derived from $TE(BAE \mid CDE)^*BT$ in the pattern in (2). $\square$

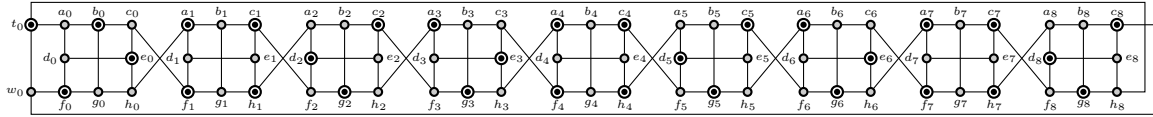


Figure 8: The minimum *IIC* set of B_8^1 with order 31.

Theorem 16. $IIC(B_n^2) = 4n + 4 - \lfloor \frac{2n}{3} \rfloor$ for $n \bmod 3 \in \{0, 1\}$.

Proof. Similar to the previous proof, the configurations that dominate and separate the vertices in $I(B_0^2)$ are shown in Figure 9, where they are denoted by **X**, **Y**, **Z**, **T**, and **W**. The second part of Table 1 displays the results regarding the configurations from Figures 5 and 9. Figures 10(a) to 10(e) shows the graph from second part of Table 1 focusing individually on the configurations $H = \{X, Y, Z, T, W\}$ for the block B_0^2 , respectively.

- The walks when $n \bmod 3 = 0, n \geq 3$:

$$\begin{aligned} TAE(BAE \mid CDE)^*BT, & \quad WDE(CDE \mid BAE)^*CW, \\ YBAE(BAE \mid CDE)^*Y, & \quad YCDE(BAE \mid CDE)^*Y, \\ ZE(BAE \mid CDE)^*BAZ, & \quad ZE(BAE \mid CDE)^*CDZ, \end{aligned} \quad (4)$$

All the walks just listed include the configuration assigned to block B_0^2 , which contributes four vertices to the set S . Additionally, in each walk, every repetition of the sequence inside the parentheses accounts for 10 vertices, and outside the parentheses there is one configuration E , and two configurations of block B , contributing an additional 10 vertices. Therefore, $|S| = \frac{10n}{3} + 4$.

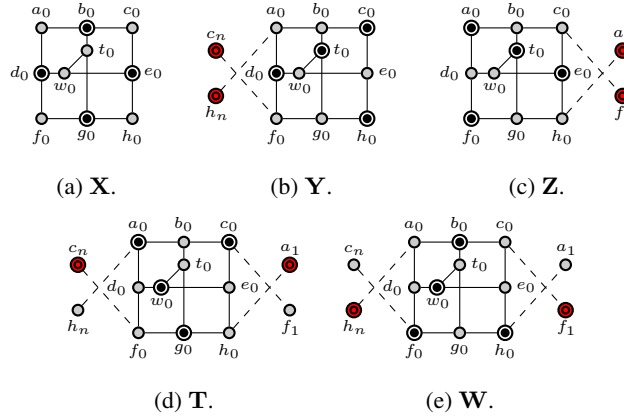


Figure 9: The possible choices for block B of the Blanuša Snark B_0^2 .

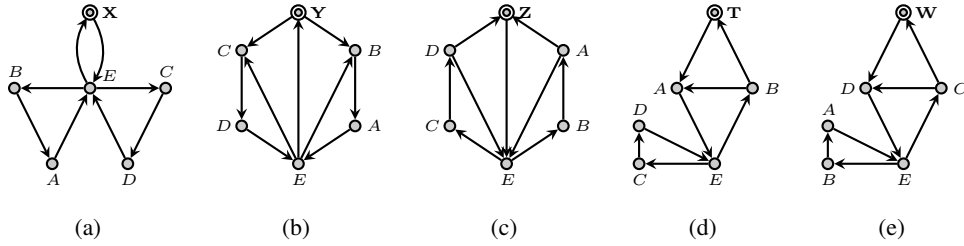


Figure 10: The transition graph of the configurations in B_n^2 .

- The walks when $n \bmod 3 = 1$:

$$XE(BAE \mid CDE)^* X. \quad (5)$$

Each repetition of the sequence inside the parenthesis accounts for 10 vertices, and an additional 4 vertices are required for the head, plus four more for the first configuration of block B . Thus, we have $|S| = \lfloor \frac{10n}{3} \rfloor + 8$. Note that $IIC(B_1^2) = 8$.

Therefore, from Equations (4) and (5), we have that $IIC(B_n^2) = \lfloor \frac{10n}{3} \rfloor + 4$, and $IIC(B_n^2) = \lfloor \frac{10n}{3} \rfloor + 8$ when $n \bmod 3$ equals 0 and 1, respectively. These expressions match those in Equation (3), and for this reason, they can be expressed equivalently, using the simplified form in Equation (3). $\square$

Corollary 17. *There is no IIC set for B_n^2 if $n \bmod 3 = 2$.*

Proof. There is no walk in any of the graphs from Figure 10 that starts at some $u \in H$, passes through configurations of block B exactly $3k + 2$ additional times, and returns to u . For this reason, there is no IIC set for B_n^2 when $n \bmod 3 = 2$. $\square$

4.3. Goldberg Snarks

Goldberg snarks are uniquely indexed by the set $\{3, 5, 7, \dots\}$. The following definition of Goldberg snarks was proposed by Campos et al. [2011]. For each such snark, the i^{th} block B_i , $i \geq 3$, odd, is the graph with vertex set $V(B_i) = \{u_i, v_i, x_i, y_i, z_i, w_i, s_i, t_i\}$, edge set $E(B_i) = \{u_i v_i, x_i y_i, x_i z_i, y_i w_i, z_i v_i, z_i t_i, v_i w_i, w_i s_i, s_i t_i\}$ and link edge set $E_{ij} = \{t_i s_j, y_i x_j, u_i u_j\}$. Figure 11(a) shows the link graph L_i , that is obtained from the basic blocks B_{i-1} and B_i , connected by

$E_{(i-1)i}$, i.e., $V(L_i) = V(B_{i-1}) \cup V(B_i)$ and $E(L_i) = E(B_{i-1}) \cup E(B_i) \cup E_{(i-1)i}$. It is evident from this definition that every Goldberg snark has an odd number of blocks.

Theorem 18. *If graph G is a Goldberg snark, then G does not have an IIC set.*

Proof. By mathematical induction. Let G_n be a Goldberg snark, where $n \geq 3$, odd. First, we show that the hypothesis holds for $n = 3$. Then we demonstrate that if the hypothesis holds for $n = k$, an arbitrary positive odd integer of at least 3, then it holds for $n = k + 2$. For contradiction, suppose that S_n is a potential IIC set of the Goldberg snark G_n .

Firstly, suppose $n = 3$. The following argument is based on the diagram of the related snark in Figure 11(b). As $C_3 = \langle u_1, u_2, u_3 \rangle$ is a 3-cycle where each of its vertices is adjacent to exactly one vertex not in C_3 , then exactly one of u_1, u_2 or u_3 is in S .

Assumption 1. W.l.o.g. suppose $u_1 \in S_3$ (and thus $u_2, u_3 \notin S_3$). For u_3 to be separated, $v_3 \in S_3$. Hence, $w_3, z_3 \notin S_3$.

Case 1. $x_1 \in S_3$. Then $y_3, z_1 \notin S_3$. For y_3 to be separated, $x_3 \in S_3$. For z_1 to be separated, $t_1 \in S_3$. As $v_1, y_1 \notin S$, $w_1 \in S_3$. For t_3 to be separated, $s_1 \in S_3$. As s_1 and t_1 are adjacent, this is a contradiction and there is no IIC set.

Case 2. $x_1 \notin S_3$. By Assumption 1, $u_1 \in S_3$ and thus, $u_2, u_3, v_1 \notin S_3$. For $u_2(u_3)$ to be separated, $v_2(v_3) \in S_3$, and hence, $w_2, z_2(w_3, z_3) \notin S_3$. In order for it to be separated, $z_1 \in S_3$, and therefore, $t_1 \notin S_3$. As $x_1, w_3 \notin S_3$, for y_3 to be separated, $y_3 \in S_3$, and thus $w_3, x_3 \notin S_3$. As $w_2, x_3 \notin S_3$, for y_2 to be separated, $y_2 \in S_3$, and so $x_2 \notin S_3$, meaning $y_1 \in S_3$ and $w_1 \notin S_3$. As $t_1, w_1 \notin S_3$, for s_1 to be separated, $s_1 \in S_3$, and $t_3 \notin S_3$. As $t_3, w_3 \notin S_3$, for s_3 to be separated, $s_3 \in S_3$, and $t_2 \notin S_3$. But, as $v_2 \in S_3$ and $t_2, x_2 \notin S_3$, it turns out that z_2 is not separated. This is a contradiction and there is no IIC set.

Secondly, let $n = k \geq 3$ be an arbitrary odd integer and suppose that G_k does not have an IIC set. Thus, G_k has the property that any potential IIC set, say S_k , is such that there exists at least one pair of vertices of S_k that are adjacent in G_k . As this property carries over to any G_{k+2} , it follows that G_{k+2} does not have an IIC set. This completes the proof. $\square$

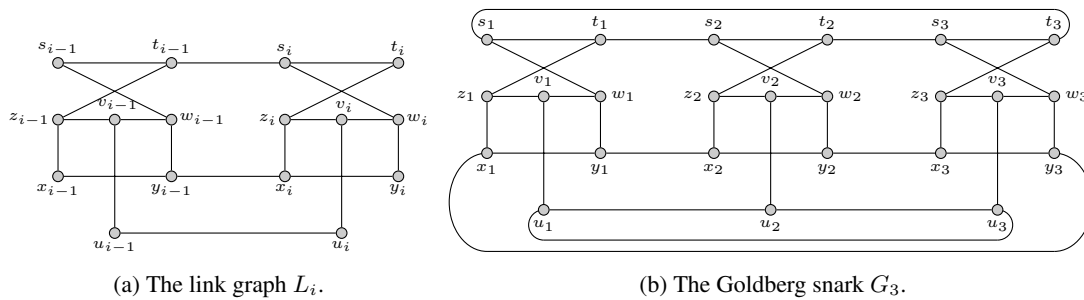


Figure 11: The link graph L_i and the Goldberg snark G_3 .

5. Conclusion

This paper reports on an investigation into the computational complexity of determining IIC sets in particular graph classes. It was proved that the problems of determining the existence of an IIC set in planar bipartite graphs or graphs with bounded degree four are $\mathcal{NP}$ -complete.

Furthermore, results for IIC sets in flower, generalized Blanuša and Goldberg snarks were provided. Given the hardness of the problem even for graphs with maximum degree four, an interesting open challenge is to determine its complexity in graphs with maximum degree three.

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