

Advances on the polyhedral study of Set Covering problems*

Leonardo Cavalcante de Abreu

Programa de Mestrado e Doutorado em Ciência da Computação
Universidade Federal do Ceará (UFC)
Campus do Pici, Bloco 910, Fortaleza, 60455-760, CE, Brazil
leonardo.abreu@lia.ufc.br

Manoel Campêlo

Departamento de Estatística e Matemática Aplicada
Universidade Federal do Ceará (UFC)
Campus do Pici, Bloco 910, Fortaleza, 60455-760, CE, Brazil
mcampelo@ufc.br

ABSTRACT

The Set Cover problem consists in, given a collection C of subsets of a finite set S and integer k , determine if there is a subset $U \subseteq C$ such that $|U| \leq k$ and every element of S belongs to at least one member of U . We study the set covering polytopes, which are defined by the convex hull of characteristic vectors of solutions to Set Cover. Set covering problems are related to structures called clutters. Clutters are collections of minimal subsets of a set. This work also extends the study of $\mathcal{N}$ -set inequalities, which are a generalization of rank inequalities. We define two classes of inequalities to covering problems and provide sufficient conditions for them to be facet-defining. The facet-defining inequalities and lifting procedures for set covering polytopes introduced in this work can be readily extended to other set covering problems.

KEYWORDS. Combinatorial optimization. Facets of polyhedra. Integer programming. Graph Theory.

OD - Discrete Optimization, TAG - Graph Theory and Related Algorithms

*Partially supported by Funcap 186-155.01.00/21 and ITR-0214-00096.01.00/23, CNPq 313153/2021-3, 312417/2022-5, 442977/2023-9 and 404479/2023-5.

1. Introduction

Covering problems are among the most important and influential problems in computer science. For instance, there are many covering problems among Karp's famous list of 21 NP-complete problems, e.g., VERTEX COVER and CLIQUE COVER [Karp, 1972].

Clutters are collections of minimal subsets of a set, and they have a strong relation with set covering problems. More specifically, a clutter (or *Sperner family*) $\mathcal{C}$ consists of an ordered pair $(V(\mathcal{C}), E(\mathcal{C}))$, such that the elements of E are subsets of V with the property that no set in E is subset of (or equal to) another set in E [Cornuéjols, 2001]. The sets V and E are respectively the set of *vertices* and *edges* of $\mathcal{C}$. A clutter is trivial if $E = \emptyset$ or $E = \{\emptyset\}$. If $e \in E(\mathcal{C})$, we write $e \in \mathcal{C}$.

If $\mathcal{C} = (V, E)$ is a clutter, a *transversal* is a set of vertices that intersects all edges of $\mathcal{C}$, while a *matching* is a set of pair-wise disjoint edges of $\mathcal{C}$. Moreover, a set of vertices is *independent* if it contains no edge of $\mathcal{C}$. An example of clutters is the set of edges of a simple graph, where a transversal is a vertex cover and a matching of the clutter is a matching of graph.

A *set covering problem* is an integer linear program of the form:

$$\min \{ \mathbf{w}^t \mathbf{x} : \mathbf{A} \mathbf{x} \geq \mathbf{e}, \mathbf{x} \in \{0, 1\}^n \}, \quad (1)$$

for some matrix $\mathbf{A} \in \{0, 1\}^{m \times n}$, $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{R}^m$ and weight vector $\mathbf{w} \in \mathbb{Q}^n$.

Let $\mathcal{C}$ be a clutter and denote by $M(\mathcal{C})$ the $\{0, 1\}$ -matrix $(m_{ij})_{|E| \times |V|}$ whose columns are indexed by $V(\mathcal{C})$ and such that $m_{ij} = 1$ if $j \in e_i$, and $m_{ij} = 0$ otherwise.

Denote by $P(\mathcal{C})$ the polytope defined by $P(\mathcal{C}) = \{ \mathbf{x} \in \mathbb{R}_+^n : M(\mathcal{C}) \mathbf{x} \geq \mathbf{e}, \mathbf{x} \leq \mathbf{e} \}$, where $\mathbf{e}$ denotes the vector of appropriate dimension whose components are all equal to 1.

Suppose now that $\mathcal{C}$ is such that $M(\mathcal{C})$ is a submatrix of $\mathbf{A}$ that can be obtained from the irredundant inequalities in $\mathbf{A} \mathbf{x} \geq \mathbf{e}$. Then (1) can be equivalently stated as follows:

$$(SC) \quad \min \{ \mathbf{w}^t \mathbf{x} : \mathbf{x} \in P(\mathcal{C}) \cap \mathbb{Z}^n \} \quad (2)$$

In this work, (2) will be called the *covering problem defined by $\mathcal{C}$* . We also denote by $P_I(\mathcal{C})$ the integer hull of $P(\mathcal{C})$. This integer polytope is what we call the *set covering polytope*. In this work, we study the facets of $P_I(\mathcal{C})$. Several facet-defining inequalities are known for this polytope [Cornuéjols and Sassano, 1989; Balas and Ng, 1989; M. Sánchez-García et al., 1998]

This work extends the study of $\mathcal{N}$ -set inequalities to set covering polytopes, which are derived from a strategy used by [Campêlo et al., 2008] to find facet-defining structures for a polytope associated to a formulation of the vertex covering problem. The name was coined by [Corrêa et al., 2019b] when studying the 2-class single group classification problem, and later used by [Araújo et al., 2023] for the geodesic classification problem. We also introduce $\mathcal{V}$ -set inequalities, which are a generalization to covering problems of the result by [Campêlo et al., 2008] and show sufficient conditions for them to be facet defining. Finally, we define lifting procedures for set covering polytopes based on variable fixing.

2. $\mathcal{N}$ -Sets

Let $\mathcal{C}$ be a clutter and let (SC) be the covering problem defined by $\mathcal{C}$. If $\tau(\mathcal{C})$ is the smallest size of a transversal, the inequality $\sum_{v \in V(\mathcal{C})} \mathbf{x}_v \geq \tau(\mathcal{C})$ is clearly valid for $P_I(\mathcal{C})$. This type of inequality is called *rank inequality*. Before stating our first result, consider the following definitions.

Definition 1. A transversal W of $\mathcal{C}$ is an $\mathcal{N}$ -set if $\tau_v = |W|$, for all $v \in W$.

We also need the concept of clutter minors and safe graphs [Corrêa et al., 2019b]. Let $\mathcal{C}$ be a clutter, then contracting a vertex $i \in V(\mathcal{C})$ is removing i from all the edges that include i , and then discarding the edges that are no longer minimal. This operation results in a clutter, denoted by $\mathcal{C}/i$, that has $V(\mathcal{C}) \setminus \{i\}$ as vertex set, and the minimal elements of $\{S \setminus \{i\} : S \in E(\mathcal{C})\}$ as edge set. Moreover, deleting a vertex i is removing all edges that include i . The resulting clutter, denoted by $\mathcal{C} \setminus i$, also has $V(\mathcal{C}) \setminus \{i\}$ as vertex set, but its edge set is $\{S \in E(\mathcal{C}) : i \notin S\}$.

Contraction and deletion operations do not depend on the order in which the operations are done, thus, denote by $\mathcal{C}/U$, resp., $\mathcal{C} \setminus U$, the clutter obtained from $\mathcal{C}$ by contracting, resp., deleting, all vertices of $U \subseteq V(\mathcal{C})$. Also, any clutter obtained from $\mathcal{C}$ by deleting or contracting vertices is called a *minor* of $\mathcal{C}$.

Definition 2. The $\mathcal{N}$ -safe graph of a clutter $\mathcal{C}$ is a simple graph, say G^{safe} , whose vertex set is $V(\mathcal{C})$ and $uv \in E(G^{\text{safe}})$ if and only if there exist two $\mathcal{N}$ -sets W_u and W_v , such that $W_u \setminus W_v = \{u\}$ and $W_v \setminus W_u = \{v\}$.

Consider a minor $\mathcal{H}$ of $\mathcal{C}$ obtained by deleting a subset of vertex $U \subseteq V(\mathcal{C})$. Denote by H^{safe} the $\mathcal{N}$ -safe graph of $\mathcal{H}$, and let $\tau_v^H = \min\{|W| : v \in W, W \in b(\mathcal{H})\}$. It follows that.

Theorem 1. Suppose that $|V(\mathcal{H})| \geq 2$. The inequality

$$\sum_{v \in V(\mathcal{H})} \frac{1}{\tau_v^H} \mathbf{x}_v \geq 1 \quad (3)$$

is valid for $P_I(\mathcal{C})$ and is facet-defining if the following conditions hold:

- i) For all $v \in U$, there is an $\mathcal{N}$ -set $W \in b(\mathcal{H})$ that is also a transversal of $\mathcal{C} \setminus (U \setminus \{v\})$.
- ii) Every connected component of H^{safe} contains an $\mathcal{N}$ -set.

One can verify that these inequalities dominate their correspondent rank inequality. Inequalities with the same rationale as (3) were introduced by [Campêlo et al., 2008] in the context of vertex coloring. They also appear in [Corrêa et al., 2019a,b] and [Araújo et al., 2023] for Euclidean and graph classification problems. These inequalities are also related to inequalities with coefficient 2 in [Balas and Ng, 1989]. The next lemma shows that the vectors that satisfy (3) with equality are exactly the characteristic vectors of the $\mathcal{N}$ -sets of $\mathcal{H}$.

Lemma 1. Let W be a transversal of $\mathcal{C}$. Then $\mathbf{x}^W$ satisfies (3) with equality if and only if $W \setminus U$ is an $\mathcal{N}$ -set of $\mathcal{H}$.

For this reason, inequalities of the same type as (3) are called $\mathcal{N}$ -set inequalities.

It is not clear whether the reciprocal of Theorem 1 is true for some class of clutters. However, if $\mathcal{C}$ has no edge with only one vertex, condition (i) is necessary.

The next result relates $\mathcal{N}$ -set inequalities to rank inequalities of minors defined by vertices with the same coefficients. To state this result, let $\mathcal{C}$ and $\mathcal{H}$ be as defined before, and let $K \subset \mathbb{N}$ be the set of all possible sizes of an $\mathcal{N}$ -set of $\mathcal{H}$, i.e., $k \in K$ if and only if there exists an $\mathcal{N}$ -set of $\mathcal{H}$ of size k . Also, let $V_k = \{v \in V(\mathcal{H}) : \tau_v^H = k\}$ and $\overline{V}_k = V(\mathcal{H}) \setminus V_k$.

Theorem 2. Assume that $\mathcal{H} = \mathcal{C} \setminus U$ satisfies condition (i) of Theorem 1. The $\mathcal{N}$ -set inequality (3) is a facet-defining inequality for $P_I(\mathcal{C})$ if and only if, for each $k \in K$, the rank inequality $\sum_{v \in V_k} \mathbf{x}_v \geq k$ is facet-defining inequality for $P_I(\mathcal{H}/\overline{V}_k)$.

3. $\mathcal{V}$ -sets

Next, we show an alternative method to strengthen rank inequalities. First, consider the following definitions, introduced here:

Definition 3. An independent set S of $\mathcal{C}$ is a $\mathcal{V}$ -set if $\alpha_v = |S|$, for all $v \in V(\mathcal{C})$.

Definition 4. The $\mathcal{V}$ -safe graph of a clutter $\mathcal{C}$ is a simple graph, say G^{safe} , whose vertex set is $V(\mathcal{C})$ and $uv \in E(G^{\text{safe}})$ if and only if there exist two $\mathcal{V}$ -sets S_u and S_v , such that $S_u \setminus S_v = \{u\}$ and $S_v \setminus S_u = \{v\}$.

Analogously to Theorem 1, let $\mathcal{H} = \mathcal{C} \setminus U$ for some $U \subseteq V(\mathcal{C})$, and $\check{H}^{\text{safe}}$ be the $\mathcal{V}$ -safe graph of $\mathcal{H}$. Let $\alpha_v^H = \max\{|S| : v \in S, \mathcal{H} \setminus \overline{S} = \emptyset\}$. Then, Theorem 3 holds:

Theorem 3. Suppose that $V(\mathcal{H})$ is not empty. The inequality

$$\sum_{v \in V(\mathcal{H})} \frac{1}{\alpha_v^H} \mathbf{x}_v \geq \sum_{v \in V(\mathcal{H})} \frac{1}{\alpha_v^H} - 1 \quad (4)$$

is valid for $P_I(\mathcal{C})$ and is facet-defining if the following conditions hold:

- i) For all $v \in U$, there is a $\mathcal{V}$ -set $S \subseteq V(\mathcal{H})$ such that $S \cup \{v\}$ is independent for $\mathcal{C}$.
- ii) All connected components of $\tilde{H}^{\text{safe}}$ contain a $\mathcal{V}$ -set.

Observe that (4) also dominates the rank inequality. Although inequalities (3) and (4) come from the same type of rank inequality, we show that they are not equivalent in most cases. One particular case when they coincide is if $\alpha_u^H = \alpha$ and $\tau_v^H = \tau$ for all $v \in V(\mathcal{H})$.

4. Lifting procedures

Let $\mathcal{C} = (V, E)$ be a clutter and denote by $P_I(\mathcal{C})$ the integer hull of $P(\mathcal{C})$. The next results derive liftings from deletion and contraction of a vertex. First, a facet-defining inequality for $P_I(\mathcal{C} \setminus v)$, for some vertex $v \in V(\mathcal{C})$, is lifted to a facet-defining inequality for $P_I(\mathcal{C})$ as follows.

Lemma 2. Let $v \in V(\mathcal{C})$ and assume that $\dim(P(\mathcal{C} \setminus v)) \geq 1$. If $\pi^t \mathbf{x} \geq \pi_0$ is facet-defining inequality of $P_I(\mathcal{C} \setminus v)$, then $\sum_{u \in V \setminus \{v\}} \pi_u \mathbf{x}_u + \alpha \mathbf{x}_v \geq \pi_0 + \alpha$ defines a facet of $P_I(\mathcal{C})$, for $\alpha = \min \left\{ \sum_{u \in V \setminus \{v\}} \pi_u \mathbf{x}_u \mid \mathbf{x} \in P_I(\mathcal{C}/v) \right\} - \pi_0$.

Analogously, a facet-defining inequality of $P_I(\mathcal{C}/v)$ can be lifted to obtain a facet-defining inequality of $P_I(\mathcal{C})$ as follows. Let $N_2(v) := \{u \in V(\mathcal{C} \setminus v) : \{u, v\} \in E\}$ denote the set of vertices that form an edge with v that has exactly two vertices. Suppose that $N_2(v) = \emptyset$.

Lemma 3. Let $v \in V$ be such that $\{v\} \notin E$ and $N_2(v) = \emptyset$, and assume $\dim(P_I(\mathcal{C}/v)) \geq 1$. If $\pi^t \mathbf{x} \geq \pi_0$ is facet-defining inequality of $P_I(\mathcal{C}/v)$, then $\sum_{u \in V \setminus \{v\}} \pi_u \mathbf{x}_u + \alpha_v \mathbf{x}_v \geq \pi_0$ defines a facet of $P_I(\mathcal{C})$, for $\alpha_v = \pi_0 - \min \left\{ \sum_{u \in V \setminus \{v\}} \pi_u \mathbf{x}_u \mid \mathbf{x} \in P_I(\mathcal{C} \setminus v) \right\}$.

We also show a similar result for the case when $N_2(v) \neq \emptyset$.

References

- Araújo, P. H., Campêlo, M., Corrêa, R. C., and Labbé, M. (2023). Integer programming models and polyhedral study for the geodesic classification problem on graphs. *European Journal of Operational Research*.
- Balas, E. and Ng, S. M. (1989). On the set covering polytope: I. all the facets with coefficients in $\{0, 1, 2\}$. *Mathematical Programming*, 43(1).
- Campêlo, M., Campos, V. A., and Corrêa, R. C. (2008). On the asymmetric representatives formulation for the vertex coloring problem. *Discrete Applied Mathematics*, 156(7).
- Cornuéjols, G. (2001). *Combinatorial Optimization: Packing and Covering*. SIAM.
- Cornuéjols, G. and Sassano, A. (1989). On the 0, 1 facets of the set covering polytope. *Mathematical Programming*, 43(1).
- Corrêa, R. C., Blaum, M., Marenco, J., Koch, I., and Mydlarz, M. (2019a). An Integer Programming Approach for the 2-class Single-group Classification Problem. *ENTCS*, 346:321–331.
- Corrêa, R. C., Delle Donne, D., and Marenco, J. (2019b). On the combinatorics of the 2-class classification problem. *Discrete Optimization*, 31:40–55.
- Karp, R. M. (1972). Reducibility among Combinatorial Problems. In *Complexity of Computer Computations: Proceedings of a symposium on the Complexity of Computer Computations*, p. 85–103. Section: 9.
- M. Sánchez-García, M.I. Sobrón, and B. Vitoriano (1998). On the set covering polytope: Facets with coefficients in $\{0, 1, 2, 3\}$. *Annals of Operations Research*, 81(0).