LVI SBPO SIMPÓSIO BRASILEIRO DE PESQUISA OPERACIONAL
DE 04 ATÉ 07 NOVEMBRO 2024
FORTALEZA | CEARÁ

Spectral Graph Theory and the Normal Modes of Coupled Oscillations

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ABSTRACT

Determining the natural frequencies of normal modes of coupled oscillations in structures is crucial to minimize the risk of ruptures. Matrix analysis of physical systems has been used for this. In this paper, we show how to model several classical physical systems using graphs and show that determining the spectrum of their Laplacian and adjacency matrices is fundamental to determining the normal modes of oscillation. From this, using results from Spectral Graph Theory, we determine the natural frequencies of several classical physical systems. On the other hand, we also show how to obtain Spectral Graph Theory results from the correspondence with physical systems.

KEYWORDS. Spectral Graph Theory, Laplacian Matrix, Eigenvalues, Coupled oscillations, Normal Modes, Natural Frequencies.

Spectral Graph Theory

DOI: 10.59254/sbpo-2024-193907
https://proceedings.science/p/193907?lang=pt-br

1. Introduction

Coupled oscillations play a fundamental role in several fields of science. Physical systems composed of two or more oscillatory elements that interact with each other can represent a diversity of natural characteristics and engineering artifacts. One of the greatest difficulties and one of the main objectives when studying this topic is the identification of the *natural frequencies* of the system, as well as its *normal modes*, which are basically oscillations in which all elements of the system are in the same frequency. Knowing that a given complex structure can oscillate at certain natural frequencies, it is possible to minimize the probability of ruptures.

Generally, university Physics books tend to present such problems of coupled oscillations separately, providing extensive particular solutions [Alonso e Finn, 1992; Nussenzveig, 2014], or not even mentioning the subject [Tipler e Mosca, 2007]. Few more advanced books present a Linear Algebra approach, involving eigenvalues and eigenvectors [Goldstein et al., 2011; Morin, 2007], but without being an approach to a general model of an oscillatory system.

In this paper, we introduce a very general model of a coupled oscillation system based on Graph Theory, allowing any number of elements in the system. We also show how natural frequencies can be obtained from the Laplacian matrix of the model graph, as well as how normal modes can be obtained from the eigenvectors of the same matrix. Therefore, several results from Spectral Graph Theory can be used for many classical coupled oscillation systems, leading to known eigenvalues and eigenvectors of certain systems with any number of blocks. To our knowledge, this approach is new in the literature.

On the other hand, we also show how to obtain results from Spectral Graph Theory from the correspondence of graphs with physical systems of coupled oscillations.

The most common example of coupled oscillation is the spring-mass system with n blocks connected by springs. We list below the main variants of this problem [Morin, 2007]. With this approach, we carried out [Cunha et al., 2024] simulations of all these systems considering any number of blocks in Desmos, a free online graphics system [Mungan, 2021; MacIsaac, 2016], which can be freely edited by any user. We know of no other simulations available on the Internet for all of these systems or for any number of blocks, much less with full access for editing by any user.

- **Spring-mass system in a circle:** n blocks on a surface, connected by springs forming a circle (Figure 1(a)).
- **Spring-mass system with no walls:** n blocks in a straight line connected by springs on a surface, without walls (Figure 1(b)).
- **Spring-mass system with 2 walls:** n blocks on a surface, with the first and last connected with springs to walls (Figure 1(c)).
- **Spring-mass system with 1 wall:** n blocks on a surface, with only the first connected with a spring to a wall (Figure 1(d)).
- **Spring-pendulum system in circle:** n pendulums with blocks connected by springs, forming a circle (Figure 1(e)).
- **Spring-pendulum system with no walls:** n pendulums with blocks connected by springs, without walls (Figure 1(f)).
- **Spring-pendulum system with 2 walls:** n pendulums with blocks connected by springs, with the first and last connected to walls (Figure 1(g)).

DOI: 10.59254/sbpo-2024-193907
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- **Spring-pendulum system with 1 wall:** n pendulums with blocks connected by springs, with only the first one connected to a wall (Figure 1(h)).

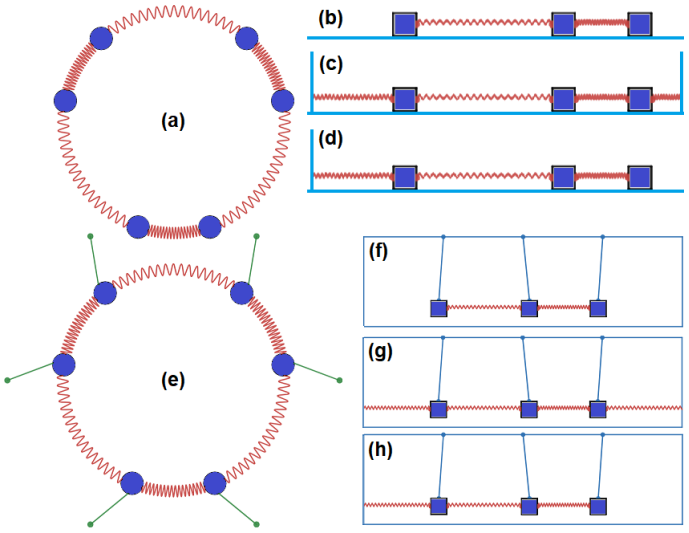

Figure 1: Examples of spring-mass systems with coupled oscillations.

Blocks are free to move with only one degree of freedom. We ignore the effects of friction or any other source of energy dissipation.

2. Matrix analysis of coupled oscillations

Next, we will use uppercase letters (Q and Z, for example) for matrices and lowercase letters with indices ($q_{i,j}$ and z_i) for matrix elements. The term *vector* will also be used to refer to $n \times 1$ column matrices. A square matrix is *diagonal* if all elements outside the main diagonal are zero. The identity matrix I_n for $n \geq 1$ is the diagonal matrix $n \times n$ with all values equal to 1 on the main diagonal. We say that a $n \times n$ square matrix Q is *diagonalizable* if there exist $n \times n$ square matrices P and D such that $Q = P \cdot D \cdot P^{-1}$, where D is a diagonal matrix and P^{-1} is the inverse of P (i.e., $P \cdot P^{-1} = I_n$). The elements of D (on the main diagonal) are called *eigenvalues* of Q and are unique. The columns of P and their multiples are *eigenvectors* of Q. More specifically, we say that a number λ is an *eigenvalue* of Q if there exists a non-zero vector X (*eigenvector* associated with λ) such that $Q \cdot X = \lambda \cdot X$.

A frequently used fact is that, if $Q = P \cdot D \cdot P^{-1}$ and $Q' = \alpha Q + \beta I_n$ for real α and β, then $Q' = P \cdot (\alpha D + \beta I_n) \cdot P^{-1}$ and, therefore, the eigenvectors of Q' are the same as those of Q and their eigenvalues can be expressed as $\lambda' = \alpha\lambda + \beta$. There are efficient algorithms to obtain the diagonalization of a matrix, that is, given the matrix Q, obtain the matrices P and D (diagonal) such that $Q = P \cdot D \cdot P^{-1}$ [Golub e Loan, 2013; Trefethen e III, 1997]. On the other hand, depending on the shape and symmetry of the matrix Q, it is possible to theoretically determine the eigenvalues and eigenvectors of Q.

We are interested in systems with equations of the type $\ddot{X} = -Q \cdot X$, where X is a column vector ($n \times 1$), $\dot{X}$ and $\ddot{X}$ are the time derivatives in first and second order (speed and acceleration), respectively, and Q is a $n \times n$ diagonalizable square matrix ($Q = P \cdot D \cdot P^{-1}$). Note that systems of this type can be rewritten as $P^{-1} \cdot \ddot{X} = -D \cdot P^{-1} \cdot X$. Taking the column vector $Z = P^{-1} \cdot X$, we have that $\ddot{Z} = P^{-1} \cdot \ddot{X}$ and therefore $\ddot{Z} = -D \cdot Z$, which consists of n equations of the type $\ddot{z}_i = -d_{i,i} \cdot z_i$, for $i = 1, \ldots, n$, since D is a diagonal matrix. This process is called *decoupling*

DOI: 10.59254/sbpo-2024-193907
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since the initial equations of the type $\ddot{x}_i = -\sum_{k=1}^{n} q_{i,k} \cdot x_k$ are "*coupled*" (i.e., the equation for $\ddot{x}_i$ does not depend only on x_i), but in the final equations of the type $\ddot{z}_i = -d_{i,i} \cdot z_i$, we have that $\ddot{z}_i$ depends only on z_i.

We will see later that, in the considered systems, the values $d_{i,i}$ of the diagonal matrix D are non-negative reals. Furthermore, we can assume that the values $d_{i,i}$ are ordered: $d_{1,1} \leq \ldots \leq d_{n,n}$. For each value $d_{i,i}$ of D (eigenvalue of Q), let $\omega_i = \sqrt{d_{i,i}}$. Therefore $\ddot{z}_i = -\omega_i^2 \cdot z_i$ is the equation of a Simple Harmonic Motion (SHM) with angular frequency ω_i, whose solution is well known, given the initial position z_i and the speed initial $\dot{z}_i$. These initial conditions can be determined from the equations $Z = P^{-1} \cdot X$ and $\dot{Z} = P^{-1} \cdot \dot{X}$, given the initial values of X and $\dot{X}$ (positions x_i and speeds $\dot{x}_i$). Once the solutions of Z are obtained, we can obtain the solutions of X by the equation $X = P \cdot Z$, that is, $x_i = \sum_{k=1}^{n} p_{i,k} \cdot z_k$ for $i = 1, \ldots, n$.

On the other hand, we can obtain the *normal modes* of oscillation by appropriately choosing the initial values of z_i and $\dot{z}_i$ for all i. In this sense, taking $z_i = a_i \cdot \sin(\omega_i t)$, where a_i is the SHM amplitude of z_i, to be defined later, we have $x_i = \sum_{k=1}^{n} p_{i,k} \cdot a_k \cdot \sin(\omega_k t)$, which consists of a linear combination of SHM movements with angular frequencies $\omega_1 \leq \ldots \leq \omega_n$, which are the *natural frequencies of the system*. This is because, given $1 \leq j \leq n$ and taking $a_k = 0$ for all k other than j, we have that $x_i = p_{i,j} \cdot a_j \cdot \sin(\omega_j t)$ for all i, causing all x_i to oscillate with the same angular frequency ω_j, thus generating a *normal mode* of the oscillatory system.

3. General model of Spectral Graph Theory

We first introduce the general coupled oscillation system model based on graphs.

Definição 3.1. *A* linear spring-mass system *has n blocks of mass m on the x axis of the frictionless Cartesian plane, linked together by springs. Blocks are numbered from 1 to n. For $i \neq j$, let $k_{i,j}$ be the spring constant between blocks i and j (let $k_{i,j} = 0$ if blocks i and j are not linked). Furthermore, for $i = 1, \ldots, n$, the block i can be connected to a "wall" on the x axis with a spring with elastic constant $k_{i,i}$ (make $k_{i,i} = 0$ if block i is not attached to a wall). Consider also that there is an* equilibrium position *in which no spring is stretched.*

Figure 2 shows a linear spring-mass system with 4 blocks and 6 springs, in which only blocks 2 and 4 connect to walls. Note that $k_{1,4} = k_{2,4} = 0$ and that $k_{1,1} = k_{3,3} = 0$.

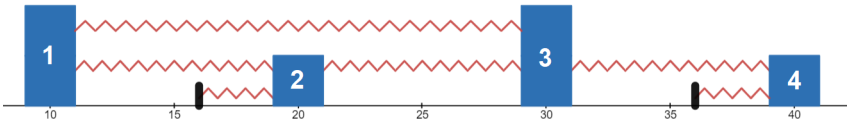

Figure 2: Example of a linear spring-mass system.

This model can be represented by a *graph*, which in this paper is defined by a set $V = \{1, \ldots, n\}$ of vertices and a set E of edges, which are non-ordered pairs of vertices. Edges from a vertex to itself are called *loops*. In a *weighted graph*, there is also a function that assigns a weight (non-negative numerical value) to each edge, including loops.

Definição 3.2. *Given a linear spring-mass system, we define the* base graph of the system *as the graph in which the blocks of the system represent vertices, the springs between blocks represent edges, and the springs from a block to a wall represent loops. Furthermore, in the* weighted graph *of the system, the edge weights are the elastic constants of each associated spring.*

Figure 3 shows the weighted graph of the linear spring-mass system in Fig. 2.

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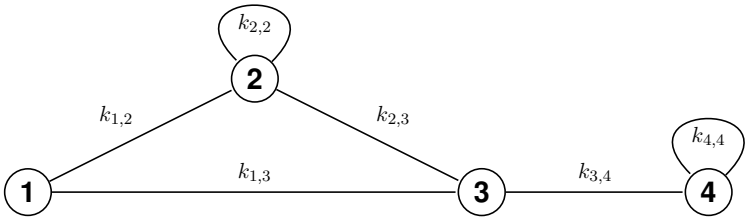

Figure 3: Weighted graph representing the linear spring-mass system in Figure 2.

There are several ways to represent a graph as a matrix. Among them, the *adjacency matrix* and the *Laplacian matrix* of the graph stand out. The *adjacency matrix* of a weighted graph is the matrix A such that $A_{i,j}$ is the weight of the edge between the vertices i and j (note that $A_{i,i} = 0$ if there is no loop at vertex i). The *Laplacian matrix* is the matrix L such that $L_{i,i}$ is the sum of the weights of the edges of the vertex i (including loops) and, for $i \neq j$, $L_{i,j}$ is -1 times the weight of the edge between vertices i and j. Note that the Laplacian matrix $L = B - A$, where B is the diagonal matrix of vertex degrees: $B_{i,i}$ is the sum of weights of the edges of the vertex i (loops are counted twice).

Let us make an important distinction here. We can represent the model using the weighted graph (with weights on the edges), as well as using just the base graph (considering weight 1 for all edges). We will use L^* and A^* for the Laplacian and adjacency matrices of the weighted graph, as well as L and A for the Laplacian and adjacency matrices of the base graph. Note that A contains only 0 and 1 and that, for all $1 \leq i \neq j \leq n$, we have that $L_{i,j} \in \{-1, 0\}$ and $L_{i,i}$ is the degree of the vertex in the base graph (number of vertex edges, where loops are counted twice).

An important fact is that Laplacian matrices are always *symmetric* and *diagonal-dominant*, that is, $L_{i,j} = L_{j,i}$ and $|L_{i,i}| \geq \sum_{j \neq i} |L_{i,j}|$ for $1 \leq i \neq j \leq n$ (same for L^*). Furthermore, in the graphs considered here, L and L^* contain only non-negative real values on the main diagonals, which implies from Gershgorin's Theorem that the Laplacian matrices of the graphs obtained from these models are *positive semidefinite* : all its n eigenvalues are real and non-negative [Salas, 1999].

Teorema 3.3. *Given a linear spring-mass system, the natural frequencies are equal to $\sqrt{\lambda_i^*/m}$, where λ_i^* are the eigenvalues of the Laplacian matrix of the system's weighted graph and m is the mass of each block. If the springs have the same elastic constant k, then the natural frequencies are equal to $\sqrt{\lambda_i \cdot k/m}$, where λ_i are the eigenvalues of the Laplacian matrix of the system's base graph.*

Proof. Let x_i be the difference between the position of block i and its equilibrium position, for all $i = 1, \ldots, n$. By Newton's 2° Law and Hooke's Law, we have that, for every block i

$$m \cdot \ddot{x_i} = -k_{i,i} \cdot x_i - \sum_{j=1}^{n} k_{i,j} \cdot (x_i - x_j) \quad \Rightarrow \quad m \cdot \ddot{x_i} = -\left(\sum_{j=1}^{n} k_{i,j} \right) \cdot x_i + \sum_{j \neq i}^{n} (k_{i,j} \cdot x_j),$$

or, in terms of the Laplacian matrix L^* of the weighted graph that represents the spring-mass system: $\ddot{x_i} = -\frac{1}{m} \cdot \sum_{j=1}^{n} L_{i,j}^* \cdot x_j$. By Section 2, the natural frequencies are equal to $\sqrt{\lambda_i^*/m}$, where λ_i^* are the eigenvalues of L^* (Laplacian matrix of the graph weighted). If all springs have elastic constant k, then $\ddot{x_i} = -\frac{k}{m} \cdot \sum_{j=1}^{n} L_{i,j} \cdot x_j$, where L is the Laplacian matrix of the base graph of the spring-mass system. Thus, the natural frequencies are equal to $\sqrt{\lambda_i \cdot k/m}$, where λ_i are the eigenvalues of L (Laplacian matrix of the base graph). □

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Corolário 3.4. *If all springs in a linear spring-mass system have elastic constant k and the base graph is d-regular, then the natural frequencies are equal to $\sqrt{(d-\lambda'_i)\cdot k/m}$, where λ'_i are the eigenvalues of the adjacency matrix of the system's base graph.*

Proof. Since the Laplacian matrix $L = B - A$, where the matrix B (of degrees) is $B = d \cdot I_n$, then the eigenvalues of L are equal to d minus each eigenvalue of A. □

4. Application for Spring-Mass Systems

In the following systems, we consider that all n blocks have the same mass m and all springs have the same elastic constant k. In each subsection, the notations L and A will refer to the Laplacian and adjacency matrices of the base graph of the system of the subsection. We will list the normal modes in ascending order of their natural frequencies. For mass-spring systems without pendulums, normal mode 1 refers to the one with the lowest positive natural frequency.

4.1. Circular spring-mass system

Initially consider the circular spring-mass system in Figure 1(a) (without walls or pendulums). This is the only system that is solved in [Morin, 2007] for any number of blocks. Only the eigenvalues (and consequently the natural frequencies) are obtained, but not the eigenvectors that characterize the normal modes of oscillation. We will give a different solution, using the general method from Section 2 and the graph model from Section 3.

At first, it may seem that this system does not fit the model described in Section 3, as the blocks are not in a straight line, but in a circle. However, it is easy to verify that this system is equivalent to the system in Figure 4 below, which fits our model.

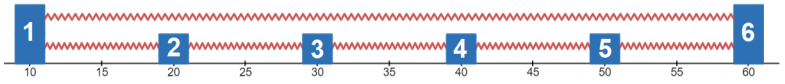

Figure 4: Representation of the circular system in Figure 1(d) according to the model in Section 3.

Teorema 4.1. *The circular spring-mass system with n blocks has $\lfloor n/2 \rfloor$ natural frequencies, which for $\ell = 1, \ldots, \lfloor n/2 \rfloor$ are equal to*

$$\omega_\ell = \sqrt{\frac{2k}{m} \cdot \left(1 - \cos\left(2\pi \cdot \frac{\ell}{n}\right)\right)},$$

Furthermore, the $n \times n$ matrix P such that $P_{j,\ell} = \sin(2\pi j\ell/n)$ contains eigenvectors (in the columns) of the system. The same for the matrix P' such that $P'_{j,\ell} = \cos(2\pi j\ell/n)$.

Proof. It follows from Theorem 3.3 and known results on eigenvalues of the Laplacian matrix of the cycle graph C_n [Hoppen et al., 2022], which is the base graph of the system. □

An interesting observation is that the greatest possible natural frequency is $2\sqrt{\frac{k}{m}}$, which only occurs when n is even. Another observation is that the natural frequency $\sqrt{3\frac{k}{m}}$ occurs for n multiple of 3, the frequency $\sqrt{2\frac{k}{m}}$ for n multiple of 4 and the frequency $\sqrt{\frac{k}{m}}$ for n multiple of 6. In short, we have the natural frequencies $\sqrt{\frac{k}{m}}$, $\sqrt{2\cdot\frac{k}{m}}$, $\sqrt{3\cdot\frac{k}{m}}$ and $2\sqrt{\frac{k}{m}}$ for n multiple of 12.

We created 2 projects in Desmos [Cunha et al., 2024] to visualize normal modes, one with a truly circular design and the other according to the equivalent model in Fig 4. Both allow any number of blocks. Figure 5 shows the 8 normal modes for the system with 16 blocks. With an additional visualization resource, highlighting the most distended springs on the dashed line, we perceive an image with ℓ "points" in the ℓ-th normal mode.

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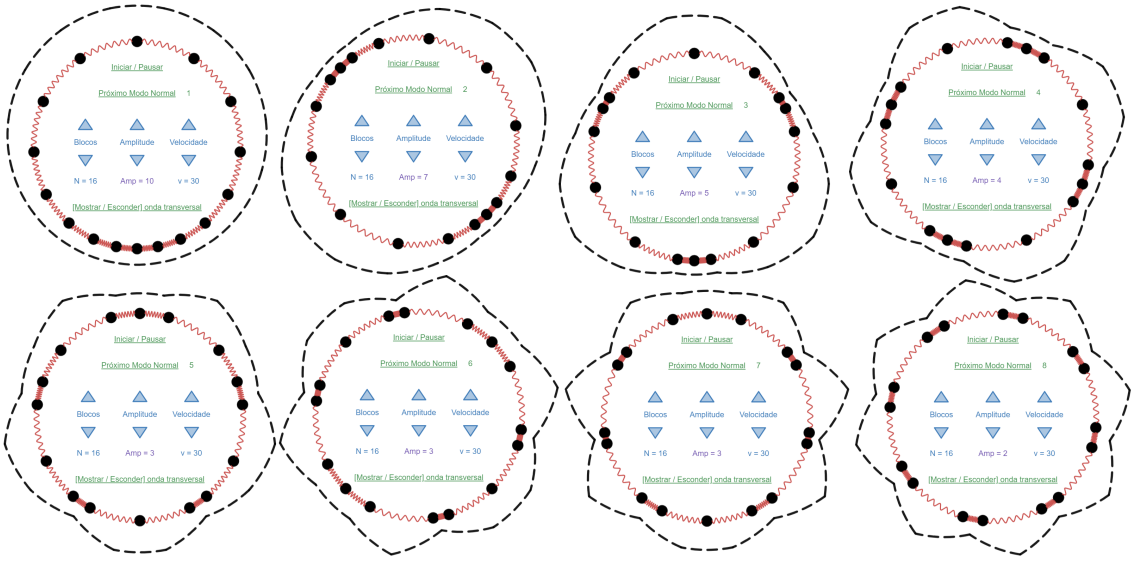

Figure 5: The 8 normal modes of the circular system with 16 blocks, shown in the Desmos simulation.

4.2. Straight spring-mass system without walls

Teorema 4.2. *The straight spring-mass system in Fig 1(a) without walls with n blocks has $n-1$ natural frequencies, which for $\ell = 1, \ldots, n-1$ are equal to*

$$\omega_\ell = \sqrt{\frac{2k}{m} \cdot \left(1 - \cos\left(\pi \cdot \frac{\ell}{n}\right)\right)}.$$

Furthermore, the $n \times n$ matrix P such that $P_{j,\ell} = \cos\left(\pi\left(j - \frac{1}{2}\right)\ell/n\right)$ contains system eigenvectors.

Proof. It follows from Theorem 3.3 and known results on eigenvalues of the Laplacian matrix of the path graph P_n [Hoppen et al., 2022], which is the base graph of the system. □

It is also possible to prove it using the physics of the system. Note that every oscillation of the straight system with n blocks implies an oscillation of the circular system with $2n$ blocks where the blocks j and $2n+1-j$ oscillate equally ($j = 1, \ldots, n$) and the springs between blocks 1 and $2n$ and blocks n and $n+1$ do not stretch. Therefore, among the n normal modes of the circular system with $2n$ blocks, only the one with the greatest frequency cannot be obtained from an oscillation of the straight system with n blocks.

Figure 6 shows the 4 normal modes of the system with $n = 5$ blocks. With an additional visualization resource, highlighting in the black line the deviation of each block from its equilibrium position, we perceive an image with $\ell + 1$ "points" in the ℓ-th normal mode, for $1\ leq\ell < n$.

An interesting observation is that, in ℓ odd modes when n is also odd (1° and 3° normal modes in Figure 6, for example), the middle block $j = (n+1)/2$ is fixed (no move), which can be proven due to the value $P_{j,\ell}$ of the eigenvector, which is always zero. This will be mentioned in Subsection 4.4 as a way to obtain the eigenvalues and eigenvectors of the system with 1 wall, given that the middle block being fixed can represent a wall.

4.3. Spring-mass system with 2 walls

Consider the spring-mass system in Figure 1(c) (2 walls, without pendulums). Note that it is represented by the graph $P_n^{(2)}$, obtained from P_n by adding 2 loops, at vertices 1 and n.

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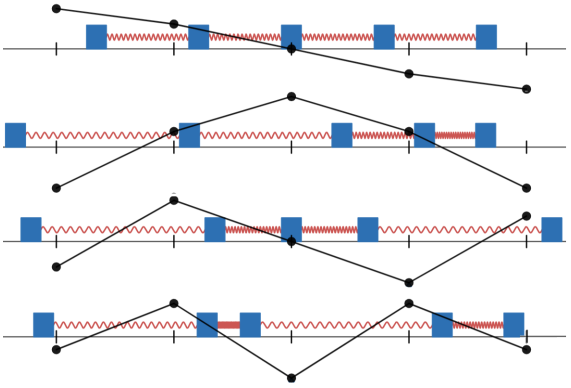

Figure 6: The 4 normal modes of the straight 5-block system without walls (Desmos simulation).

Teorema 4.3. *The straight spring-mass system with 2 walls with n blocks has n natural frequencies, which for $\ell = 1, \ldots, n$ are equal to*

$$\omega_\ell \;=\; \sqrt{\frac{2k}{m} \cdot \left(1 - \cos\left(\pi \cdot \frac{\ell}{n+1}\right)\right)}.$$

Furthermore, the $n \times n$ matrix P such that $P_{j,\ell} = \sin\left(\pi j \ell/(n+1)\right)$ contains system eigenvectors (in the columns).

Proof. An interesting fact is that the n natural frequencies of this system of n blocks with 2 walls are the same as the system of $n+1$ blocks without walls, seen in the previous section. In other words, the eigenvalues of the Laplacian of $P_n^{(2)}$ are the same as those of P_{n+1} with the exception of zero. This is because it is possible to associate the elongation (or deformation) of each of the $n+1$ springs of the system of n blocks with 2 walls to the deflection of each of the n blocks of a system without walls from their equilibrium positions.

To verify this, let y_j be the elongation (or deformation) of spring j, where the springs are numbered from 0 to n, with spring 0 before block 1 and spring j after block j for $j = 1, \ldots, n$. Therefore $y_0 = x_1$, $y_n = -x_n$ and $y_j = x_{j+1} - x_j$. Thus, for $1 < j < n$,

$$\ddot{y_0} \;=\; \ddot{x}_1 \;= -\frac{k}{m} \cdot (2x_1 - x_2) \;= -\frac{k}{m} \cdot (y_0 - y_1)$$

$$\ddot{y_n} \;=\; -\ddot{x}_n \;= +\frac{k}{m} \cdot (2x_n - x_{n-1}) \;= -\frac{k}{m} \cdot (y_n - y_{n-1})$$

$$\ddot{y_j} \;=\; -\frac{k}{m} \cdot (2y_j - y_{j-1} - y_{j+1})$$

Note that these $n+1$ equations in y regarding the spring elongations are the same equations in x of the block deviations in the system with n blocks without walls, from the previous section. Therefore, the natural frequencies of the springs are the same as in the previous section, as mentioned. The eigenvalues of the Laplacian of $P_n^{(2)}$ are $2 \cdot (1 - \cos(\pi\ell/(n+1)))$ for $1 \le \ell \le n$ and the natrual frequencies are $\omega_\ell = \sqrt{\frac{2k}{m} \cdot \left(1 - \cos\left(\pi \cdot \frac{\ell}{n+1}\right)\right)}$. Note also that $x_j = \sum_{b=0}^{j} y_b$ for $j = 1, \ldots, n$.

Thus, for each eigenvalue $\lambda > 0$ with eigenvector v of the Laplacian of P_{n+1}, we have the same eigenvalue $\lambda > 0$ for the Laplacian $P_n^{(2)}$ with eigenvector v' such that $v'_j = \sum_{b=1}^{j} v_b$ for

DOI: 10.59254/sbpo-2024-193907
https://proceedings.science/p/193907?lang=pt-br

$j = 1, \ldots, n$. That is, the ℓ-th eigenvector of the Laplacian of $P_n^{(2)}$ is the vector v' such that

$$v'_j = \sum_{b=1}^{j} \cos\left(\frac{\pi\ell}{n+1} \cdot \left(b - \frac{1}{2}\right)\right).$$

Using Euler's formula for complex numbers $e^{i\theta} = cos\theta + i \cdot \sin\theta$, we have the following geometric progression sum, since the real part $Re(e^{i\theta}) = \cos\theta$:

$$v'_j = Re\left(\sum_{b=1}^{j} \exp\left(i\frac{\pi\ell}{n+1}\left(b - \frac{1}{2}\right)\right)\right) = \frac{\cos\left(\frac{\pi\ell}{n+1}\left(j - \frac{1}{2}\right)\right) - \cos\left(\frac{\pi\ell}{n+1}\left(j + \frac{1}{2}\right)\right)}{2 - 2\cos\left(\frac{\pi\ell}{n+1}\right)}$$

$$\Rightarrow \quad v'_j = \frac{2\sin\left(\frac{\pi j\ell}{n+1}\right)\sin\left(\frac{\pi\ell/2}{n+1}\right)}{2 - 2\cos\left(\frac{\pi\ell}{n+1}\right)}$$

Disregarding multiplicative constants for fixed ℓ, we obtain the eigenvector v'' below for the ℓ-th eigenvalue: $v''_j = \sin\left(\frac{\pi j\ell}{n+1}\right)$. □

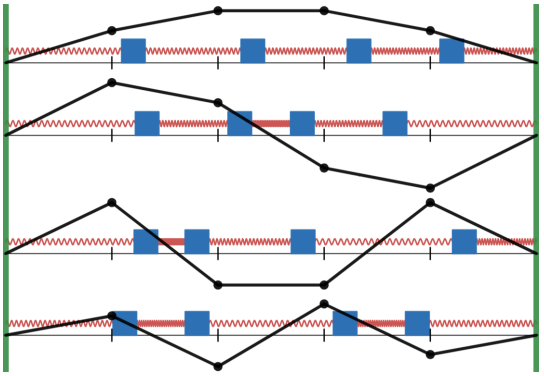

Figure 7: The 4 normal modes of the straight system of 4 blocks with 2 walls (Desmos simulation).

Figure 7 shows the 4 normal modes of the 4-block system with 2 walls. With an additional visualization resource, highlighting in the black line the deviation of each block from its equilibrium position, we perceive an image with ℓ "points" in the ℓ-th normal mode.

Furthermore, in ℓ odd modes when n is even (1° and 3° normal modes in Fig 7, for example), the middle spring, between the blocks j and $j+1$ for $j = \frac{n}{2}$ never stretch, which can be proven since $v_j = v_{j+1}$ always in this case. This fact will be used in Subsection 4.4 to obtain the eigenvalues and eigenvectors of the system with 1 wall, given that, when the middle spring does not stretch, it is as if we had two independent systems with 1 wall and half of the blocks.

An important point is that the proof of the previous theorem used physical arguments to obtain natural frequencies. With this physical arguments, we also obtain Spectral Graph Theory results: the eigenvalues and eigenvectors of the Laplacian matrix of the graph class $P_n^{(2)}$.

Corolário 4.4. *The Laplacian matrix of the graph $P_n^{(2)}$ has eigenvalues $1 - \cos(\pi\ell/(n+1))$ for $n \geq 2$ and $\ell = 1, \ldots, n$. Furthermore, the $n \times n$ matrix P such that $P_{j,\ell} = \sin(\pi j\ell/n)$ contains eigenvectors of the Laplacian matrix of the graph $P_n^{(2)}$.*

DOI: 10.59254/sbpo-2024-193907
https://proceedings.science/p/193907?lang=pt-br

4.4. Spring-mass system with 1 wall

Teorema 4.5. *The straight spring-mass system in Fig 1(d), with 1 wall and n blocks, has n natural frequencies, which for $\ell = 1, \ldots, n$ are equal to*

$$\omega_\ell = \sqrt{\frac{2k}{m} \cdot \left(1 - \cos\left(\pi \cdot \frac{2\ell - 1}{2n+1}\right)\right)},$$

Furthermore, the $n \times n$ matrix P such that $P_{j,\ell} = \sin\left(\pi j(2\ell-1)/(2n+1)\right)$ contains system eigenvectors.

Proof. It is easy to see that this system is represented by the graph $P_n^{(1)}$, which is the base graph obtained from the path P_n by adding 1 loop at vertex 1. An interesting fact is that the n natural frequencies of this system of n blocks with 1 wall are contained in the $2n$ natural frequencies of the system of $2n$ blocks with 2 walls. This is because, for each normal mode of the first, we can obtain a normal mode of the second, matching the oscillations so that the middle spring (between blocks n and $n+1$) does not stretch. Another interesting fact is that the n natural frequencies of the system of n blocks with 1 wall are contained in the $2n$ natural frequencies of the system of $2n+1$ blocks without walls. This is because, for each normal mode of the first, we can obtain a normal mode of the second, making the central block $(n+1)$ remain fixed and both the n blocks on the left and the n blocks on the right oscillate equally as in the system of n blocks with 1 wall. These two facts are illustrated in Figure 8: a normal mode of the 3-block system with 1 wall occurring both in the 6-block system with 2 walls and in the 7-block system without walls.

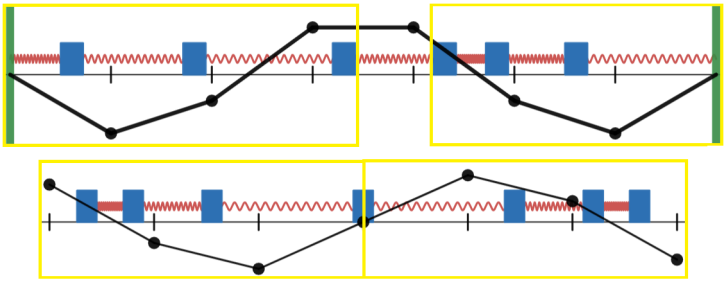

Figure 8: Normal mode of 3 blocks with 1 wall (yellow rectangles) appearing in both the 6-block system with 2 walls and the 7-block system without walls.

In order to determine their values, note the eigenvectors of the system of $2n$ blocks with 2 walls (Section 3.3) in which the n-th coordinate is equal to $(n+1)$-th coordinate (blocks n and $n+1$ with the same oscillation). We saw in Section 3.3 the eigenvalue $2 \cdot (1 - \cos(\pi\ell'/(2n+1)))$ and the eigenvector v'' with values $v''_j = \sin(\pi j\ell'/(2n+1))$ for each $\ell' = 1, \ldots, 2n$. It is easy to see that $v''_n = v''_{n+1}$ if and only if ℓ' is odd. Thus, for $\ell = 1, \ldots, n$ and taking $\ell' = 2\ell - 1$ odd, we have for the Laplacian matrix of $P_n^{(1)}$ the eigenvalue $2 \cdot \left(1 - \cos\left(\pi \cdot \frac{2\ell-1}{2n+1}\right)\right)$ and the eigenvector v with values $v_j = \sin\left(\pi j \cdot \frac{2\ell-1}{2n+1}\right)$. Therefore, we have for $\ell = 1, \ldots, n$ the natural frequencies of the theorem.

Identical results would be obtained alternatively analyzing the system of $2n+1$ blocks without walls. Simulations were carried out in Desmos, making it possible to compare how the normal modes of the system of n blocks with 1 wall are actually associated with the odd normal modes of the system of $2n$ blocks with 2 walls and the system of $2n+1$ blocks without walls. □

DOI: 10.59254/sbpo-2024-193907
https://proceedings.science/p/193907?lang=pt-br

As in the previous section, we also obtain Spectral Graph Theory results from the physical oscillation arguments: the eigenvalues and eigenvectors of the Laplacian matrix of the graph class $P_n^{(1)}$.

Corolário 4.6. *The Laplacian matrix of the graph $P_n^{(1)}$ has eigenvalues $1-\cos(\pi(2\ell-1)/(2n+1))$ for $n \geq 2$ and $\ell = 1, \ldots, n$. Furthermore, the $n \times n$ matrix P such that $P_{j,\ell} = \sin(\pi j(2\ell-1)/(2n+1))$ contains eigenvectors of the Laplacian matrix of the graph $P_n^{(1)}$.*

4.5. Spring-mass systems with pendulums

Now consider the systems in Fig 1 with pendulums, where we assume that they all have the same length ℓ. With small oscillations, we can consider that the blocks are on the x-axis of the Cartesian plane (with no contact to the ground). When the pendulum i has a very small angle θ_i with the vertical, the component of the weight in the direction of movement is always contrary to the movement, with value $m \cdot g \cdot \sin\theta_i \approx m \cdot g \cdot x_i/\ell$. As a result, all equations of the system in question over $\ddot{x}_i$ will have a factor $-g \cdot x_i/\ell$ compared to the corresponding systems without pendulums, from the previous sections. We can also fit these systems with pendulums into the graph model seen in Section 3, making each block connected to a wall by a spring with elastic constant $k' = m \cdot g/\ell$. Thus, systems with pendulums can be represented by weighted graphs, where each vertex has an additional loop with weight $m \cdot g/\ell$ and the other edges have weight k (from the springs). Therefore, the eigenvectors of systems with pendulums are the same as those of the corresponding systems without pendulums and the eigenvalues are the same plus g/ℓ. For example, for an eigenvalue λ of a system without pendulums, we have the natural frequency $\sqrt{\frac{k}{m} \cdot \lambda}$, while we have the natural frequency $\sqrt{\frac{g}{\ell} + \frac{k}{m} \cdot \lambda}$ in the corresponding system with pendulums.

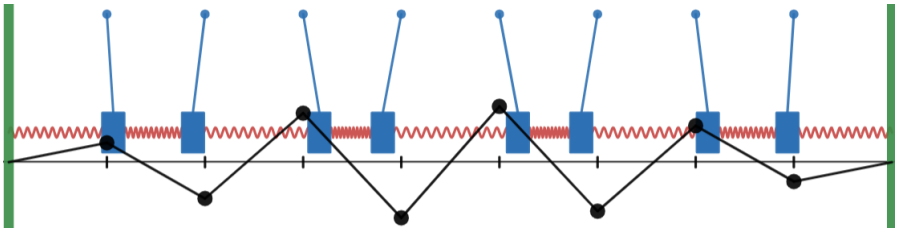

Figure 9: Greatest natural frequency of normal mode of the straight 8-block pendulum system with 2 walls, shown in Desmos: the 8-th normal mode.

5. Applications for vibrating string systems

Now consider the system of a string with n tied blocks separated by a distance ℓ (Figure 10), on the frictionless horizontal plane. This system has similar variants to the spring-mass systems, depending on whether the ends of the string are fixed at a point or free on a straight perpendicular line. Considering small oscillations, we can assume that the string tension in the blocks is always the same, whose value will be denoted by T.

Therefore: $m_i \cdot \ddot{x}_i = -T\sin\theta_{i-1} + T\sin\theta_i$ for $1 < i < n$. As $\sin\theta_i = (x_{i+1} - x_i)/\ell$, we have the following, assuming all masses equal: $\ddot{x}_i = -\frac{T}{m \cdot ell} \cdot \Big(2x_i - x_{i-1} - x_{i+1}\Big)$. If the first end is fixed, then: $m_1 \cdot \ddot{x}_1 = -T\sin\theta_0 + T\sin\theta_1$. Since $\sin\theta_0 = x_1/\ell$ and $\sin\theta_1 = (x_2 - x_1)/\ell$, then $\ddot{x}_1 = -\frac{T}{m \cdot \ell} \cdot \Big(2x_1 - x_2\Big)$. Otherwise, $m_1 \cdot \ddot{x}_1 = T\sin\theta_1$ and therefore $\ddot{x}_1 = -\frac{T}{m \cdot \ell} \cdot \Big(x_1 - x_2\Big)$.

Similarly for the last block. With this, we can model this system and its variants with the same matrices (including eigenvalues and eigenvectors) as the spring-mass systems in Sections 4.2, 4.3, 4.4 . Desmos simulations of these spring-mass systems already show simulations for the corresponding string systems, in the option "*Show transverse wave*".

DOI: 10.59254/sbpo-2024-193907
https://proceedings.science/p/193907?lang=pt-br

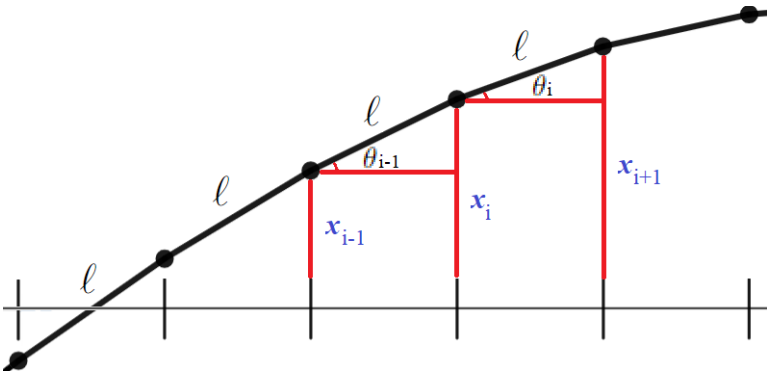

Figure 10: String system with n equidistant blocks with the same mass m.

References

Alonso, M. e Finn, E. J. (1992). *Physics*. Addison-Wesley.

Cunha, F., Guilhon, I., Madeiro, S., Sampaio, A. J., e Sampaio, R. (2024). Coupled Oscillations Normal Modes in Desmos. link: `https://github.com/rudinims/normal_modes`.

Goldstein, H., Poole, C. P., e Safko, J. (2011). *Classical Mechanics*. Pearson, 3 edition.

Golub, G. H. e Loan, C. F. V. (2013). *Matrix Computations*. Johns Hopkins University Press.

Hoppen, C., Jacobs, D. P., e Trevisan, V. (2022). *Locating Eigenvalues in Graphs*. Springer.

MacIsaac, D. (2016). Desmos: A free cross-platform calculating and graphing tool. *The Physics Teacher*, 54(8):509–509.

Morin, D. (2007). *Introduction to Classical Mechanics*. Cambridge University Press.

Mungan, C. E. (2021). Using desmos to understand the difference between phase and group velocity. *The Physics Teacher*, 59(1):30–33.

Nussenzveig, M. (2014). *Curso de Física Básica (Vol. 2)*. Editora Blucher, São Paulo, Brasil.

Salas, H. N. (1999). Gershgorin's theorem for matrices of operators. *Linear algebra and its applications*, 291(1-3):15–36.

Tipler, P. A. e Mosca, G. (2007). *Physics for Scientists and Engineers*. W. H. Freeman.

Trefethen, L. N. e III, D. B. (1997). *Numerical Linear Algebra*. SIAM, Philadelphia, PA, USA.

DOI: 10.59254/sbpo-2024-193907
https://proceedings.science/p/193907?lang=pt-br