

A multiple allocation p-hub location problem in a Competitive Scenario: a second order conic formulation

Tainá Pôssas Abreu

Centro Federal de Educação Tecnológica de Minas Gerais (CEFET-MG)
Av. Amazonas 5253 - Nova Suíça - Belo Horizonte - MG - Brasil
tainapossas@cefetmg.br

Ricardo S. de Camargo

Universidade Federal de Minas Gerais
Av. Antônio Carlos, 6627, Pampulha - Belo Horizonte - MG - Brasil
rcamargo@dep.ufmg.br

Fátima M. de Souza Lima

Universidade Federal de Minas Gerais
Av. Antônio Carlos, 6627, Pampulha - Belo Horizonte - MG - Brasil
fatimamslima@face.ufmg.br

Luiza Bernardes Real

Instituto Federal de Minas Gerais
Av. Michel Pereira de Souza, 3007 - Campinho, Congonhas - MG - Brasil
luizabernardesreal@gmail.com

Débora Alves Ribeiro

Instituto Federal de Minas Gerais
Cidade Administrativa Presidente Tancredo Neves, Rodovia Papa João Paulo II, 4143 - Belo Horizonte - MG - Brasil
deboralvesribeiro@gmail.com

ABSTRACT

This work presents a new second order conic formulation for a multiple allocation in a competitive scenario of p-hub location problem. We optimize the decision making process of a newly airline or entrant to establish its hub-and-spoke network amid competition. We propose a non-linear optimization formulation that instead of maximizing the capture of market share, it minimizes the lost market share. We model our problem as a mixed integer second order conic program (MISOCP), which can be solved by modern off-the-shelf solvers. Through extensive computational experiments, we show that our formulation consistently outperforms the existing approach of the literature on solving benchmark instances in terms of computational running times.

KEYWORDS. Hub location. competition. second order conic program. non-linear optimization model.

Paper topics (L&T, OA)

1. Introduction

In this study, a new model for a competitive hub location problems (CHL) for the airline industry is proposed. As the increasing need for air transportation requires better planning tools, we devised a model for the case of a newly established airline, known as an entrant, that desires to establish its hub-and-spoke (HS) network in a competitive market having other established companies. The market share captured through the location of hubs plays a crucial role in the network efficiency, serving as a key driver in reducing operational costs, improving transportation times, and improving customer satisfaction. Strategically positioned hubs act as distribution and consolidation centers, facilitating the flow of goods/passengers.

The main objective of this paper is to provide a simpler, more robust and efficient model for the CHL problem when compared with the one proposed by [Tiwari et al. \[2021b\]](#), which maximizes the market share of an airline company. Our formulation is derived from a non-linear integer program (NLIP), which still poses as a computational challenge for standard solvers such as CPLEX, Gurobi, and others. Nevertheless, we can rely on a mixed integer second order conic approach to ease the solution process for these solvers.

In our work, we formulate the problem of a newcomer company in a non-cooperative multiplayer scenario, using the probability of choice based on the utility of the alternatives. Usually, most hub location problems seek the minimizing of cost functions [\[Drezner and Eiselt, 2023\]](#), that often produce solutions that overestimate the demand using the hub network. Nonetheless, the real demand is normally affected by existing competition. Hence, the need to model competitive assumptions has a great economic appeal, being focus of this work.

The location of hubs and the design of transportation networks have long been attracted the attention of the scientific community, since these underline structures induce great economic savings given that they take advantage of scale economies in hub-to-hub connections [\[O'Kelly, 1986, 1987\]](#). Location models for competitive systems have been discussed by the research community for more than 95 years [\[Hotelling, 1929\]](#). However, the loss of demand only began to be explored in 2006 by [Berman et al. \[2006\]](#). All competitive location models share a common thread that emphasizes the interconnectedness of four key variables: demand, distance, attractiveness of the facility, and market share. In particular, the first three variables are considered independent factors, while the market share is considered as an outcome variable [\[Drezner, 2019\]](#). Our work explores a multiple allocation problem under a scenario of competition, considering the loss market share, yet to be better studied.

Introduced by [O'Kelly \[1986\]](#), hub location problems have been extensively studied especially when related to areas of air transportation networks and telecommunication systems. The hubs act as central structures responsible for shifting flows, instead of directly connect origin-destination (OD) pairs [\[O'Kelly, 1987\]](#). Hubs serve as crucial intermediate points in the routing process, but are also used for switching, transshipment, sorting, and distribution processes [\[de Camargo et al., 2009\]](#). Please, refer to the surveys of [Campbell and O'Kelly \[2012\]](#), [Farahani et al. \[2013\]](#), [Contreras and O'Kelly \[2019\]](#) and [Alumur \[2019\]](#) for more extensive reviews.

We assume the use of one or at most two intermediate hubs in a route to service origin-destination pair of exchanging demand nodes. The economies of scale are represented as a constant discount factor (α) in inter-hub connections as proposed in [Tiwari et al. \[2021b\]](#). We further vary this inter-hub discount to observe the robustness of the model. We highlight that our model outperforms the one of [Tiwari et al. \[2021b\]](#)'s in 93.33% of the 120 tested instances. We solved problems with 10, 15, 20 and 25 nodes, and achieved speed ups of 3.4, 3.8, 2.9 and 4.3 times in our model, respectively. The perceived speed up of the instances we lost were marginal, i.e., we lost by a marginal amount in only 8 instances.

The remainder of this work is organized as follows. Section 2 presents an literature review over the last five years on competitive hub location. Section 3 formally defines the problem and introduces the existing literature model and ours to solve it. Section 4 shows how the second order conic approach can be used to both formulations. Section 5 reports the results of the numerical experiments. Section 6 highlights the final considerations and discusses next steps of the research.

2. Literature Review

The competitive location problem involves determining optimal locations for one or more hubs or facilities while considering existing competitors. The hubs are expected to service as points for aggregating and reorganizing demand, which is affected by the attractiveness of the hubs in the power of choice of alternatives. The goal is to maximize the captured market share by the new facilities, since is proportional to market share.

The recent review of [Drezner and Eiselt \[2023\]](#) branches the strategy of locating facilities in a competitive scenario in two approaches. The first is based on game theory, which incorporates competitor reactions into decision-making models. Such models empowers decision-makers with a comprehensive understanding of market dynamics. The other evolves location models with deterministic single-objective optimization problems with the assumption that competitors will not respond immediately to the decisions made by the new entrant. Our work is located in the latter one.

[Marianov et al. \[1999\]](#) dealt with the following situation. A small airline aims to attract passenger traffic from larger companies by capitalizing on the assumption that these companies optimize their total costs based on very high traffic figures, indicating a significant market share. Consequently, there must be a portion of customers who are under serviced, experiencing longer travel times than necessary due to sub-optimal service. [Drezner and Drezner \[2001\]](#) applied the attraction function to model such assumptions for hub location. They used an algorithm based on gravity theory that states that a passenger have to make a choice of hubs to pass before going to the destination node. Please, refer to [Drezner \[2019\]](#) for greater detail on gravity models in competitive facility location and [Eiselt et al. \[2019\]](#) to better understand on consumer behavior.

The work of [Eiselt and Marianov \[2009\]](#) merges with the ones of [Marianov et al. \[1999\]](#) and [Drezner and Drezner \[2001\]](#). They model an entrant's competitive hub location problem involving decisions of the locations of hub facilities and choosing paths to route the demand flows between each origin-destination pair through one and at most two intermediate hubs. Their problem is solved by means of a heuristic concentration method. [Tiwari et al. \[2021b\]](#) extended the problem by allowing only a single path to route the flows between an origin-destination pair. However, we have found some non-conformities in their results. Their source code sets the scale economy parameter as an integer-valued variable instead of being set as a float-type one. This mistake led them to obtain the same solutions for the same size of instances but different values for the scale economies. Here, we have addressed these non-conformities while comparing our results with theirs.

As stated in the most recent review of [Drezner and Eiselt \[2023\]](#) on competitive location models, the field of study continues to progress, but yet lacks a definitive method to solve all problems. Existing models rely on numerous assumptions, but properly modeling customer behavior remains a challenge.

3. Problem description and notation

The problem concerns the location of p-hubs with multiple allocation in a competitive market. Let N be the set of nodes representing airports, where each node can act as both a hub candidate and a demand origin. Define $W = \{(i, j) \in N \times N : i \neq j\}$ as the set of origin-destination nodes that exchange $f_{ij} \geq 0$ units of flow, $(i, j) \in W$.

In the competitive scenario under consideration, we denote a set C of airlines operating in a hub-and-spoke structure, competing for their respective market shares, along with a single emerging airline e aiming to enter the market. Each airline $a \in C \cup \{e\}$ manages its own set of hubs $H^a = \{(k, l) \in \{N \times N\}\}$ and routes its captured demand through one or, at most, two of its hubs to take advantage of inter-hub scale economies.

To estimate the market share, one can use the probability of choosing a path within the hub-and-spoke network given the utilities of the paths [Eiselt and Marianov, 2009; Tiwari et al., 2021a,b]. The utility u_{iklj}^a of a path $i \rightarrow k \rightarrow l \rightarrow j$ to be used by the users of origin in $i \in N$ going to $j \in N$ (pair $(i, j) \in W$), while using airline $a \in C \cup \{e\}$, through hubs $(k, l) \in H^a$ can be modeled as the gravity function such as:

$$u_{iklj}^a = \frac{A_{kl}^a}{\gamma(T_{iklj}^a)^\beta + (1 - \gamma)(B_{iklj}^a)^\delta} \quad (1)$$

where A_{kl}^a represents the attractiveness of inter hub connection (k, l) used to route demand $i - j$, and B_{iklj}^a and T_{iklj}^a represent the cost and total time of the flight, respectively. The parameters β and δ signify the attraction decay associated with travel time and cost, respectively. The parameter $\gamma \in [0, 1]$ allows the trade off between travel time and travel costs.

So, the proportion of the market share captured by the entry airline (ρ^e) is defined as the following function, for each origin-destination pair:

$$\rho_{ij}^e = \frac{\sum_{(k,l) \in H^e} u_{iklj}^e x_{iklj}^e}{\sum_{(k,l) \in H^e} u_{iklj}^e x_{iklj}^e + \sum_{c \in C} \sum_{(k,l) \in H^c} u_{ik,lj}^c}, \quad \forall (i, j) \in W \quad (2)$$

where x_{iklj}^e in the numerator represents the selected paths used by the entrant, and in the denominator there are all available paths, including those chosen by the competitors $u_{ik,lj}^c$ originated from solving the uncapacitated multiple allocation p-hub location problem.

The demand captured by the entrant using a path $i \rightarrow k \rightarrow l \rightarrow j$ is equal to the product of the total demand between the airports and the above probability of the market share or $f_{ij} \rho_{iklj}^e$. So, to maximize its market share by installing a predefined number of p hubs in the network, the company must find the optimal set of locations. However, instead of studying the maximization of the captured market, we propose an approach that minimizes the loss of market share, as proposed by Lin and Tian [2021], for a facility location problem.

In this paper, we compare the results of our minimization model with the formulation of Tiwari et al. [2021b]. We adopted the notation and decision variables from Table 1 to present our formulation and that of Tiwari et al. [2021b]. Theirs can be written as follows:

$$\max \quad \theta_{MA}(x) = \sum_{(i,j) \in W} f_{ij} \left\{ \frac{\sum_{(k,l) \in H^e} u_{iklj}^e x_{iklj}^e}{\sum_{(k',l') \in H^e} u_{ik',l'j}^e x_{ik',l'j}^e + \sum_{c \in C} \sum_{(k',l') \in H^c} u_{ik',l'j}^c} \right\} \quad (3)$$

$$s.t. : \sum_{k \in H} y_k = p \quad (4)$$

$$\sum_{l \in H} \sum_{k \in H} x_{iklj} = 1, \quad \forall (i, j) \in W \quad (5)$$

$$\sum_{l \in H} x_{iklj} + \sum_{l \in H \setminus k} x_{ilkj} \leq y_k, \quad \forall (i, j) \in W, \quad k \in N \quad (6)$$

$$y_k \in \{0, 1\}, \quad \forall k \in N \quad (7)$$

$$x_{iklj} \in \{0, 1\}, \quad \forall (i, j) \in W, \quad (k, l) \in H \quad (8)$$

The objective function (3) maximizes the capture of market share. Constraint (4) enforces p hubs to be open. Constraints (5) state that all demand flow should be routed through one or at most two hubs. Constraints (6) are the coupling constraints. Constraints (7) and (8) show the domain of the variables y and x , respectively. Formulation (3)-(8) is a nonlinear program, which its current form cannot be handle by off-the-shelf solvers.

3.1. Our formulation

To set our second order conic formulation, we know that the captured market share plus the lost part is equal to one. So, the objective function (3) is equivalent to:

$$\min \sum_{(i,j) \in W} f_{ij} \left\{ 1 - \frac{\sum_{(k,l) \in H} u_{iklj} x_{iklj}}{\sum_{(k',l') \in H} u_{ik'l'j} x_{ik'l'j} + \sum_{(k',l') \in H^c} u_{ik'l'j}^c} \right\} \quad (9)$$

Normalizing the utilities by the competitors, we have:

$$\min \sum_{(i,j) \in W} f_{ij} \left\{ 1 - \frac{\sum_{(k,l) \in H} \frac{u_{iklj}}{u_{iklj}^c} x_{iklj}}{\sum_{(k,l) \in H} \frac{u_{iklj}}{u_{iklj}^c} x_{iklj} + \sum_{(k,l) \in H^c} \frac{u_{iklj}^c}{u_{iklj}^c}} \right\} \quad (10)$$

Assuming that,

$$\pi_{iklj} = \frac{u_{iklj}}{u_{iklj}^c} \quad (11)$$

Tabela 1: The adopted notation and the list of decision variables

Sets

N : Set of nodes. $|N| = n$.

H^a : Set of candidate hub pairs for an airline $a \in \{e\} \cup C$. $H^a = \{(k, l) \in \{N \times N\}\}$.

W : Set of origin-destination pairs. $W = \{(i, j) \in \{N \times N\} : i \neq j\}$.

Parameters

p : Number of hubs.

f_{ij} : Total demand of the origin-destination pair $(i, j) \in W$.

u_{iklj}^a : The utility of the path $i \rightarrow k \rightarrow l \rightarrow j$ using any airline $a \in \{e\} \cup \{c\}$; equals to 0 if the path does not exist.

Decision Variables

$x_{iklj} \in \{0, 1\}$: 1 if nodes k and l are hubs used by pair (i, j) , 0 otherwise.

$y_k \in \{0, 1\}$: 1 if node k is a hub for the entrant, 0 otherwise.

we finally have:

$$\min \sum_{(i,j) \in W} f_{ij} \left\{ 1 - \frac{\sum_{(k,l) \in H} \pi_{iklj} x_{iklj}}{\sum_{(k,l) \in H} \pi_{iklj} x_{iklj} + 1} \right\} \quad (12)$$

After simplifying, we can write our formulation as:

$$\min \theta(y_k, x_{iklj}) = \sum_{(i,j) \in W} f_{ij} \left\{ \frac{1}{\sum_{(k,l) \in H} \pi_{iklj} x_{iklj} + 1} \right\} \quad (13)$$

$s.t. : (4) - (8)$

where the objective function (13) minimize the loss of the market share for the entrant airline.

It is worth noting that we can relax the binary variables x_{iklj} to assume continuous values within the range $[0, 1]$. This can be justified by the following rationale: when both y_k and y_l , which are binary variables themselves, are set to 1, x_{iklj} will invariably assume a value of 1. Conversely, when either y_k and y_l equals 0, x_{iklj} takes on a value of 0. Consequently, it is possible to relax equation (8) as $x_{iklj} \in [0, 1]$.

4. Mixed-integer second order conic program

A mixed-integer second order conic program is a special type of modeling that, though still yielding non-linear programs, allows one to use off-the-shelf solvers to tackle optimization problems, as shown in [Atamtürk et al. \[2012\]](#). Applying their guidelines, [Tiwarei et al. \[2021b\]](#)'s problem can be converted into:

$$\max \sum_{(i,j) \in W} f_{ij} V_{ij} \quad (14)$$

$$s.t. : (4) - (7)$$

$$x_{iklj} \in [0, 1] \quad \forall (i, j) \in W, (k, l) \in H \quad (15)$$

$$V_{ij} + \sum_{c \in C} \sum_{(k,l) \in H^c} u_{iklj}^c x_{iklj} Q_{ij} = 1 \quad \forall (i, j) \in W \quad (16)$$

$$r_{ij} = \sum_{(k,l) \in H^e} u_{iklj}^e x_{iklj} + \sum_{c \in C} \sum_{(k,l) \in H^c} u_{iklj}^c \quad \forall (i, j) \in W \quad (17)$$

$$(V_{ij} + r_{ij})^2 \geq 2 \sum_{(k,l) \in H^e} u_{iklj}^e x_{iklj}^2 + V_{ij}^2 + r_{ij}^2 \quad \forall (i, j) \in W \quad (18)$$

$$(Q_{ij} + r_{ij})^2 \geq 2 + Q_{ij}^2 + r_{ij}^2 \quad \forall (i, j) \in W \quad (19)$$

$$Q_{ij}, V_{ij}, r_{ij} \geq 0 \quad \forall (i, j) \in W \quad (20)$$

with $Q_{ij} = \frac{1}{\sum_{(k,l) \in H^e} u_{iklj}^e x_{iklj} + \sum_{c \in C} \sum_{(k,l) \in H^c} u_{iklj}^c}$ and $V_{ij} = Q_{ij} \sum_{(k,l) \in H^e} u_{iklj}^e x_{iklj}$. Both rela-

tions allow us to obtain the constraints (16). We also have $V_{ij} = \frac{\sum_{(k,l) \in H^e} u_{iklj}^e x_{iklj}}{r_{ij}}$, where r_{ij} is

equal to the denominator of the objective function (3), as can be seen in constraints (17). These result into $V_{ij}r_{ij} \geq \sum_{(k,l) \in H^e} u_{iklj}^e x_{iklj}$, which can be written as the rotated second order conic

constraints (18). Likewise $Q_{ij} = \frac{1}{r_{ij}}$ can be written as $Q_{ij} r_{ij} \geq 1$, which can also be rewritten as the rotated second order conic constraints (19).

It can be observed that the objective function is linear and the constraints are linear or conic quadratic. It should be observed that as z_{ij} is greater than 1, γ_{ij} is at most equals to 1.

5. Computational experiments and performance

We compares our formulation with one of [Tiwari et al. \[2021b\]](#). The computational experiments used the same datasets, commonly studied by the HLP literature: the Civil Aeronautics Board (CAB) dataset, which comprises data on passenger flow volumes and distances among 25 US cities. It includes problems with 10, 15, 20 and 25 nodes. The computation of customers' utility information will be the same presented in their work.

All computational tests were conducted on a ASUS VivoBook workstation, equipped with an 11th Gen Intel(R) Core(TM) i7-1165G7 @ 2.80GHz and 20 GB of RAM, running on a 64-bit Linux environment. We coded the methods in C++ and use CPLEX Optimization Studio 22.1.1, configured for sequential operation, meaning processing on only one thread. We use as the default termination criterion an optimality gap of 0.01% or a maximum CPU time limit of 1 hour.

The computational performances are shown in Table 2, where solutions for our model and theirs were obtained using different combinations of p and α in the correspond CAB dataset. We can see that our model outperforms [Tiwari et al. \[2021b\]](#)'s in 93.33% of the 120 tested instances. It can be seen by Table 3 that we achieved speed ups of 3.4, 3.8, 2.9 and 4.3 times in our model, respectively for instances with 10, 15, 20 and 25 nodes. The perceived speed up of the instances we lost were marginal. All running times geometric means were lower in our model, with emphasis on the 25 nodes instance.

To highlight the difference in computational time between **both of the** approaches, we study their probabilistic guarantees by plotting their performance profiles [\[Dolan and Moré, 2002\]](#) from the computational experiments. If $t_{p,s}$ denotes the CPU time taken to solve instance p using solution approach s , then the performance ratio $r_{p,s}$ can be calculated as $r_{p,s} = \frac{t_{p,s}}{\min_{s \in S} t_{p,s}}$.

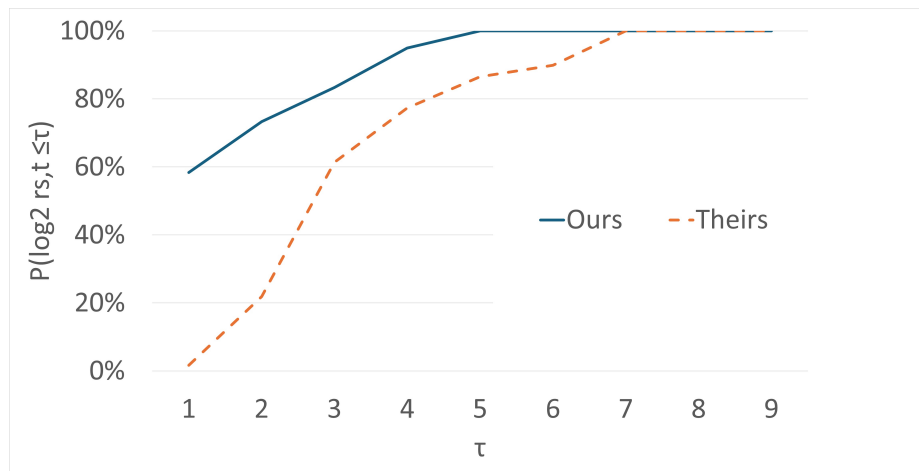


Figura 1: Performance profile of both formulations

In order to compare the models performance, we use the ratio of the probability that

a given formulation has a performance ratio $r_{p,s}$ within the limit τ , as $\rho_s(\tau) = P(r_{p,s} \leq 2^\tau)$. In another words, the performance ratio represents the likelihood that the CPU time for a given approach (s) is no more than 2^τ times the CPU time of the best approach.

The performance profile in Figure 1 highlights the superiority of our model that solves all instances of the problem before $\tau = 5$, while the reference's models finishes all instances only at $6 \leq \tau \leq 7$.

6. Conclusion and final remarks

In conclusion, our proposed model offers a simpler, more robust, and efficient approach to the problem, outperforming the model presented by Tiwari et al. [2021b] in terms of computational time. Our modeling approach of minimizing the loss of market share proved to be a significant advancement over traditional assumption. Future work could explore the more advanced approaches to solve the problem as can be done with Generalized Benders Decomposition [de Camargo et al., 2009] or Outer Approximation with Benders Decomposition methods [De Camargo et al., 2011]. Additionally, investigating sources of uncertainty in the problem data, such as transportation flows could make the model more realistic and adaptable to practical situations.

Referências

- Alumur, S. A. (2019). Hub location and related models. *Contributions to Location Analysis: In Honor of Zvi Drezner's 75th Birthday*, p. 237–252.
- Atamtürk, A., Berenguer, G., and Shen, Z.-J. (2012). A conic integer programming approach to stochastic joint location-inventory problems. *Operations Research*, 60(2):366–381.
- Berman, O., Krass, D., and Wang, J. (2006). Locating service facilities to reduce lost demand. *IIE Transactions*, 38(11):933–946.
- Campbell, J. F. and O'Kelly, M. E. (2012). Twenty-five years of hub location research. *Transportation Science*, 46(2):153–169.
- Contreras, I. and O'Kelly, M. (2019). Hub location problems. *Location science*, p. 327–363.
- De Camargo, R. S., de Miranda Jr, G., and Ferreira, R. P. (2011). A hybrid outer-approximation/benders decomposition algorithm for the single allocation hub location problem under congestion. *Operations Research Letters*, 39(5):329–337.
- de Camargo, R. S., Miranda Jr, G., Ferreira, R. P. M., and Luna, H. (2009). Multiple allocation hub-and-spoke network design under hub congestion. *Computers & Operations Research*, 36(12):3097–3106.
- Dolan, E. D. and Moré, J. J. (2002). Benchmarking optimization software with performance profiles. *Mathematical programming*, 91:201–213.
- Drezner, T. (2019). Gravity models in competitive facility location. *Contributions to Location Analysis: In Honor of Zvi Drezner's 75th Birthday*, p. 253–275.
- Drezner, T. and Drezner, Z. (2001). A note on applying the gravity rule to the airline hub problem. *Journal of Regional Science*, 41(1):67–72.
- Drezner, Z. and Eiselt, H. (2023). Competitive location models: A review. *European Journal of Operational Research*.

- Eiselt, H. A. and Marianov, V. (2009). A conditional p-hub location problem with attraction functions. *Computers & Operations Research*, 36(12):3128–3135.
- Eiselt, H. A., Marianov, V., and Drezner, T. (2019). *Competitive location models*. Springer.
- Farahani, R. Z., Hekmatfar, M., Arabani, A. B., and Nikbakhsh, E. (2013). Hub location problems: A review of models, classification, solution techniques, and applications. *Computers & industrial engineering*, 64(4):1096–1109.
- Hotelling, H. (1929). Stability in competition, kecon. *The Economic Journal*, Vol. 39, No. 153: 41–57.
- Lin, Y. H. and Tian, Q. (2021). Branch-and-cut approach based on generalized benders decomposition for facility location with limited choice rule. *European Journal of Operational Research*, 293(1):109–119.
- Marianov, V., Serra, D., and ReVelle, C. (1999). Location of hubs in a competitive environment. *European journal of operational research*, 114(2):363–371.
- O’kelly, M. E. (1986). The location of interacting hub facilities. *Transportation science*, 20(2): 92–106.
- O’kelly, M. E. (1987). A quadratic integer program for the location of interacting hub facilities. *European journal of operational research*, 32(3):393–404.
- Tiwari, R., Jayaswal, S., and Sinha, A. (2021a). Alternate solution approaches for competitive hub location problems. *European Journal of Operational Research*, 290:68–80. ISSN 03772217.
- Tiwari, R., Jayaswal, S., and Sinha, A. (2021b). Competitive hub location problem: Model and solution approaches. *Transportation Research Part B: Methodological*, 146:237–261. ISSN 01912615.

Tabela 2: Computational performance for both formulations corresponding to all CAB datasets

N	p	α	Competitor's hubs	Market Share (%)	Our	Theirs	N	p	α	Competitor's hubs	Market Share (%)	Our	Theirs
					CPU(s) (gap %)	CPU(s) (gap %)						CPU(s) (gap %)	CPU(s) (gap %)
10	2	0,1	[4,5]	54,19	1,23	8,07	20	2	0,1	[13,19]	52,30	5,24	11,80
		0,2	[4,5]	54,26	1,53	5,11			0,2	[13,19]	52,35	3,66	11,65
		0,3	[4,5]	54,35	0,95	5,26			0,3	[13,19]	52,05	4,91	10,93
		0,4	[4,5]	54,44	1,17	5,01			0,4	[13,19]	51,79	4,29	11,82
		0,5	[5,10]	52,38	0,83	4,37 (0,01)			0,5	[13,19]	51,84	5,19	10,75
		0,6	[5,10]	52,46	1,16	4,20			0,6	[13,19]	51,57	4,37	10,49
		0,7	[5,10]	52,53	1,05	5,16			0,7	[13,19]	51,12	6,74	10,50
		0,8	[5,10]	52,60	1,86	8,84			0,8	[13,19]	51,15	12,25	10,23
		0,9	[5,10]	51,83	1,29	6,91			0,9	[13,19]	50,94	6,06	11,00
		1	[5,10]	51,45	0,90	5,42			1	[13,19]	50,69	2,72	10,29
10	3	0,1	[4,5,10]	55,45	0,81	6,41	20	3	0,1	[4,13,19]	53,22	3,36	9,18
		0,2	[4,5,10]	55,62	1,00	6,80			0,2	[4,13,19]	53,28	1,95	9,99
		0,3	[4,5,10]	55,76	1,25	5,25			0,3	[4,13,19]	53,03	2,45	9,05
		0,4	[4,5,10]	55,95	1,35	6,23			0,4	[4,13,19]	52,82	2,95	9,60
		0,5	[4,5,10]	55,55	1,07	6,39			0,5	[4,13,19]	52,90	2,13	9,40
		0,6	[4,5,10]	55,48	2,06	8,73			0,6	[4,13,19]	52,69	2,17	11,78
		0,7	[4,5,10]	55,63	1,38	5,70			0,7	[4,13,19]	52,29	1,82	7,95
		0,8	[2,8,10]	51,96	1,59	3,27			0,8	[5,13,19]	51,32	2,57	8,53
		0,9	[2,8,10]	52,06	1,55	2,96			0,9	[5,13,19]	51,13	1,77	8,87
		1	[5,8,10]	50,88	1,30	7,27			1	[5,13,19]	50,63	1,19	7,26
10	4	0,1	[3,4,5,8]	55,68	0,10	0,22	20	4	0,1	[4,13,17,19]	53,34	125,58	562,09
		0,2	[1,4,5,10]	56,17	0,08	0,19			0,2	[4,13,17,19]	53,47	126,64 (0,01)	218,21 (0,01)
		0,3	[1,4,5,10]	56,37	0,09	0,18			0,3	[4,13,17,19]	53,28	123,22	401,27 (0,01)
		0,4	[2,4,8,10]	54,23	0,08	0,21			0,4	[4,13,17,19]	53,13	103,10	425,47
		0,5	[2,4,8,10]	54,47	0,74	1,06			0,5	[4,13,17,19]	53,27	118,81	271,96
		0,6	[2,4,8,10]	54,03	0,85	0,20			0,6	[4,13,17,19]	53,12	127,40	231,09
		0,7	[4,5,8,10]	53,15	0,08	0,21			0,7	[4,13,17,19]	53,07	136,31	211,30
		0,8	[4,5,8,10]	53,30	0,10	0,20			0,8	[5,13,17,19]	52,14	32,95	238,89 (0,01)
		0,9	[2,5,8,10]	52,35	0,10	0,20			0,9	[5,13,17,19]	51,99	120,06	210,80
		1	[2,5,8,10]	51,22	0,12	0,33			1	[5,11,13,19]	50,72	79,74	91,01
15	2	0,1	[5,13]	53,35	0,91	1,79	25	2	0,1	[13,19]	52,11	515,77	-
		0,2	[5,13]	52,08	0,08	1,97			0,2	[13,19]	52,15	532,39	4687,11
		0,3	[5,13]	51,45	0,51	1,92			0,3	[13,19]	51,95	538,73	227,25
		0,4	[5,13]	51,51	0,70	1,83			0,4	[13,19]	51,71	506,07	4471,60
		0,5	[5,13]	50,65	0,44	1,83			0,5	[13,19]	51,76	450,14	4201,96
		0,6	[5,13]	50,69	0,54	1,99			0,6	[19,24]	51,15	343,30	4269,35
		0,7	[5,13]	50,72	0,76	1,76			0,7	[19,24]	51,18	370,90	4542,95
		0,8	[5,13]	50,76	0,65	1,84			0,8	[19,24]	51,21	500,06	4483,95
		0,9	[5,13]	50,79	0,66	1,96			0,9	[19,24]	51,25	59,72	4631,93
		1	[5,13]	50,73	0,83	1,87			1	[19,24]	51,05	268,71	3781,31
15	3	0,1	[4,5,13]	54,37	14,68	90,45 (0,01)	25	3	0,1	[4,19,24]	53,45	43,88	33,54
		0,2	[4,5,13]	53,21	13,32	95,49			0,2	[4,19,24]	53,35	43,20	38,18
		0,3	[4,5,13]	52,64	19,98	46,15			0,3	[4,19,24]	53,41	10,36	34,18
		0,4	[4,5,13]	52,69	14,65	97,57			0,4	[4,19,24]	53,29	8,16	34,23
		0,5	[4,5,13]	51,92	12,57	89,53			0,5	[4,19,24]	53,36	8,11	31,86
		0,6	[4,5,13]	52,00	14,75	95,64			0,6	[13,19,24]	51,70	12,05	34,48
		0,7	[4,5,13]	52,07	17,63	94,58			0,7	[13,19,24]	51,47	8,01	36,69
		0,8	[5,8,13]	51,44	15,64	46,47			0,8	[13,19,24]	51,50	9,16	34,84
		0,9	[5,8,13]	51,38	9,46	51,20			0,9	[13,19,24]	51,31	47,65	36,05
		1	[5,8,13]	51,13	12,13	19,92			1	[13,19,24]	51,04	15,01	32,87
15	4	0,1	[4,5,12,13]	54,98	17,80 (0,01)	61,52	25	4	0,1	[4,17,19,24]	54,90	528,50 (0,01)	1500,56
		0,2	[4,5,12,13]	53,54	10,88 (0,01)	25,73			0,2	[4,17,19,24]	54,61	49,21	2061,75
		0,3	[4,5,12,13]	53,79	17,45	71,97			0,3	[4,17,19,24]	54,71	26,48	1690,73
		0,4	[4,5,8,13]	52,84	16,52	38,71			0,4	[4,17,19,24]	54,63	436,02	1327,16
		0,5	[4,5,8,13]	52,09	11,61 (0,01)	20,87 (0,01)			0,5	[4,13,19,24]	52,75	20,87	1523,23
		0,6	[4,5,8,13]	52,22	12,00	35,16			0,6	[4,13,19,24]	52,74	348,74	1545,66
		0,7	[4,5,8,13]	52,29	13,15	62,76			0,7	[4,13,19,24]	52,55	421,84	748,51
		0,8	[4,5,8,13]	52,42	13,89	58,05			0,8	[2,13,19,24]	51,56	411,44	64,20
		0,9	[5,8,9,13]	52,00	12,06	36,91			0,9	[2,13,19,24]	51,43	407,53 (0,01)	50,79
		1	[5,8,11,13]	51,18	14,18	69,19			1	[2,13,19,24]	51,11	215,65 (0,01)	1210,44

Tabela 3: Geometric mean of speed up and running times of both formulations

	N = 10	N = 15	N = 20	N = 25	Global
speed up g.m.	3,4	3,8	2,9	4,3	3,6
our running time g.m.	0,6	4,7	10,4	99,8	7,4
their running time g.m.	2,0	17,6	29,5	403,3	24,8