

An asymmetric formulation to the harmonious coloring problem

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ABSTRACT

A proper k -coloring of a graph G is a *harmonious coloring* if each pair of distinct edges have different sets of colors in their endpoints. The *harmonious chromatic number*, denoted by $h(G)$, is the minimum $k \in \mathbb{N}$ such that G has a harmonious k -coloring. In this paper, we present an exact integer formulation denoted *asymmetric representative formulation* and give some computational results for it on random graphs with distinct edge densities.

KEYWORDS. graph coloring. representative formulation. operational research.

1. Introduction

Given a simple graph $G = (V, E)$, a k -coloring of G is a function $c : V(G) \rightarrow \{1, \dots, k\}$. A k -coloring c is *proper* if $c(u) \neq c(v)$, for every $uv \in E(G)$. We say that a k -coloring c of G is *harmonious*, defined with this terminology at [Lee and Mitchem, 1987], if c is a proper coloring such that for any pair of distinct edges $uv, xy \in E(G)$, we have that $\{c(u), c(v)\} \neq \{c(x), c(y)\}$. A coloring satisfying only this last constraint, without being necessarily proper, is also known as a *line-distinguishing coloring*. The *harmonious chromatic number* of G is $h(G) = \min\{k \in \mathbb{N} \mid G \text{ admits a harmonious } k\text{-coloring}\}$.

In our work (to appear) we define and present computational experiments for three new exact integer formulations for the harmonious coloring problem: the standard model, the representative formulation [Campêlo et al., 2004] and the asymmetric representative formulation. Here we introduce only the asymmetric formulation and we show some experiments carried out for this formulation.

To the best of our knowledge, the only work in the literature to present formulations to a similar coloring problem is the one by [de Oliveira, 2019]. However, in this work, the author does not consider the restriction of the coloring to be proper, as only line-distinguishing colorings are addressed.

2. Asymmetric Representative Formulation for Harmonious Coloring

A harmonious k -coloring of G can be viewed as a partition $\{S_1, \dots, S_k\}$ of $V(G)$ into k of independent sets such that, for any $1 \leq i \neq j \leq k$, there is at most one edge with one endpoint in S_i and the other in S_j . In particular, any two vertices u, v in the same independent set (i.e., having the same color) have no common neighbor.

Suppose that, for each color $i \in \{1, \dots, k\}$, we choose a vertex to be the representative of the color class S_i . Then the vertices can be in one of two states: it represents the color or there exists another vertex that represents its color. Notice that each vertex v can represent itself and also all vertices that are non-adjacent to v , nor share a common neighbor with v . The use of this concept in graphs gives some symmetric solutions when dealing with integer programming formulations as any vertex of a given color can be chosen as its representative. In order to tackle this problem, we assume a total ordering over the vertex set $V(G)$ represented in the sequel as the relation $\leq$. Then, we define $N_+^*(u) = \{v \in \overline{N}(u) \mid N(v) \cap N(u) = \emptyset \text{ and } u \leq v\}$ and $N_-^*(u) = \{v \in \overline{N}(u) \mid N(v) \cap N(u) = \emptyset; v \leq u\}$. Thus, we define the binary variable x_{uv} , for every $u \in V(G)$ and $v \in N_+^*(u)$ and x_{uu} for every $u \in V(G)$, meaning whether or not the vertex u is the representative of the color assigned to the vertex v . Over such variables, we can define the following formulation.

The objective function (1a) below is just what one might expect, since each representative represents a different color. Constraints (1b) indicate that each vertex $v \in V(G)$ must be represented either by itself or by some vertex smaller than itself (w.r.t the ordering we considered) that indeed can receive the same color than v , i.e. some vertex in $N_-^*(v)$. Constraints (1c) indicate that each pair of adjacent vertices, that can be represented by the same vertex, must be assigned distinct colors. Constraints (1d) indicates that, given two vertices u and w , if they have the same color (have the same representative z), then there must at most one vertex l in their conjoint neighborhood that has a particular color (represented in this case by vertex v); also, notice that if they do not share a color, in their conjoint neighborhood any color appears only twice. The Constraint (1e) forces that two vertices that share a neighbor cannot have the same representative, allowing the line-distinguishing coloring of the graph. The Constraint (1f) forces that only a representative can represent the color of a vertex. Finally, Constraints (1g) are the definition of the variable's bounds.

$$(ARFHC) \min \sum_{v \in V} x_{vv} \quad (1a)$$

$$\text{s.t.} \quad \sum_{u \in (N_+^*(v) \cup \{v\})} x_{uv} \geq 1 \quad v \in V(G), \quad (1b)$$

$$x_{vu} + x_{vw} \leq x_{vv} \quad u, w \in N_+^*(v), uw \in E(G), \quad (1c)$$

$$\sum_{l \in N_+^*(v) \cap N(\{u, w\})} x_{vl} \leq 3 - x_{zu} - x_{zw} \quad v \in V(G), uw \in E(G), z \in N_+^*(\{u, w\}), \quad (1d)$$

$$\sum_{z \in N_+^*(v) \cap N(u)} x_{vz} \leq x_{vv} \quad u, v \in V(G), \quad (1e)$$

$$x_{vu} \leq x_{vv} \quad v \in V(G), u \in N_+^*(v), \quad (1f)$$

$$x_{vu} \in \{0, 1\} \quad v \in V(G), u \in N_+^*(v) \quad (1g)$$

2.1. Computational Experiments for (ARFHC)

All computational experiments were executed on a machine running Ubuntu 20.04 x86-64 GNU/Linux, with an Intel Core i7-7500 Octa-Core 2.70 GHz processor and 20Gb of RAM. The ILP formulation was implemented in Julia and solved with Gurobi 10.0.1.

A preliminary round of computational experiments empirically showed that the asymmetric formulation is the most effective among the formulations proposed in our work. In this context, we tested the asymmetric formulation on a set of instances named `random`. In this set, for a given number of vertices n and a probability p we generate a graph with n vertices and add an edge between every pair of distinct vertices with probability p . Note that such a process is inherently random and can produce different structures, thus we generate for each pair of vertex number n and edge probability p , 5 graphs with the same parameters in order to avoid particular structures.

We set a time limit of 1 hour for the solver to execute the formulation with the standard configuration and do not restrict the number of threads used by the solver. The asymmetric formulation was able to solve almost all of the instances presented in the random set, as reported in Table 1. Furthermore, this formulation was able to solve optimally even the instances with 100 vertices in less than 2 seconds for densities bigger than 10%.

Nevertheless, the formulation was not able to optimally solve any instance with a density smaller than or equal to 10% and more than 30 vertices. Furthermore, all instances with density 5% and more than 40 vertices reach the time limit of 1800 seconds and present an optimality gap that increases with the number of vertices upon 70%. These experiments suggest that challenging instances should have a lower density, which means a small number of edges.

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$ V(G) $	%	Sol	Gap	Time	$ V(G) $	%	Sol	Gap	Time
10	0.05	3.0	0.0	0.001	60	0.05	16.6	36.9	TL
10	0.10	4.8	0.0	0.001	60	0.10	24.8	25.8	TL
10	0.20	5.2	0.0	0.002	60	0.20	40.6	0.0	0.211
10	0.30	7.2	0.0	0.001	60	0.30	56.4	0.0	0.001
10	0.40	7.8	0.0	0.001	60	0.40	60.0	0.0	0.001
10	0.50	9.0	0.0	0.001	60	0.50	60.0	0.0	0.001
10	0.60	9.8	0.0	0.0	60	0.60	60.0	0.0	0.001
10	0.70	10.0	0.0	0.0	60	0.70	60.0	0.0	0.001
10	0.80	10.0	0.0	0.0	60	0.80	60.0	0.0	0.001
10	0.90	10.0	0.0	0.0	60	0.90	60.0	0.0	0.001
20	0.05	5.6	0.0	0.071	70	0.05	20.4	46.0	TL
20	0.10	7.4	0.0	0.081	70	0.10	30.2	35.3	TL
20	0.20	10.8	0.0	0.041	70	0.20	48.0	0.0	0.295
20	0.30	14.4	0.0	0.003	70	0.30	68.4	0.0	0.001
20	0.40	19.0	0.0	0.001	70	0.40	70.0	0.0	0.001
20	0.50	19.6	0.0	0.001	70	0.50	70.0	0.0	0.001
20	0.60	20.0	0.0	0.0	70	0.60	70.0	0.0	0.001
20	0.70	20.0	0.0	0.0	70	0.70	70.0	0.0	0.001
20	0.80	20.0	0.0	0.0	70	0.80	70.0	0.0	0.001
20	0.90	20.0	0.0	0.0	70	0.90	70.0	0.0	0.001
30	0.05	9.0	0.0	101.553	80	0.05	26.0	56.1	TL
30	0.10	11.0	0.0	5.325	80	0.10	35.6	39.9	TL
30	0.20	16.4	0.0	0.67	80	0.20	58.0	0.0	0.101
30	0.30	23.4	0.0	0.002	80	0.30	78.2	0.0	0.001
30	0.40	29.2	0.0	0.001	80	0.40	80.0	0.0	0.001
30	0.50	30.0	0.0	0.0	80	0.50	80.0	0.0	0.001
30	0.60	30.0	0.0	0.0	80	0.60	80.0	0.0	0.001
30	0.70	30.0	0.0	0.0	80	0.70	80.0	0.0	0.001
30	0.80	30.0	0.0	0.0	80	0.80	80.0	0.0	0.001
30	0.90	30.0	0.0	0.0	80	0.90	80.0	0.0	0.001
40	0.05	10.6	13.3	TL	90	0.05	30.0	53.9	TL
40	0.10	14.6	7.3	1096.094	90	0.10	41.4	36.3	TL
40	0.20	22.8	0.0	1.979	90	0.20	69.6	0.0	0.011
40	0.30	34.6	0.0	0.001	90	0.30	89.8	0.0	0.001
40	0.40	39.6	0.0	0.001	90	0.40	90.0	0.0	0.001
40	0.50	40.0	0.0	0.001	90	0.50	90.0	0.0	0.001
40	0.60	40.0	0.0	0.001	90	0.60	90.0	0.0	0.001
40	0.70	40.0	0.0	0.0	90	0.70	90.0	0.0	0.001
40	0.80	40.0	0.0	0.0	90	0.80	90.0	0.0	0.001
40	0.90	40.0	0.0	0.001	90	0.90	90.0	0.0	0.001
50	0.05	13.8	24.7	TL	100	0.05	33.6	70.1	TL
50	0.10	19.4	10.8	1248.21	100	0.10	48.6	34.5	TL
50	0.20	29.6	0.0	0.963	100	0.20	81.4	0.0	0.027
50	0.30	45.2	0.0	0.001	100	0.30	99.4	0.0	0.001
50	0.40	49.6	0.0	0.001	100	0.40	100.0	0.0	0.001
50	0.50	50.0	0.0	0.001	100	0.50	100.0	0.0	0.001
50	0.60	50.0	0.0	0.001	100	0.60	100.0	0.0	0.001
50	0.70	50.0	0.0	0.001	100	0.70	100.0	0.0	0.001
50	0.80	50.0	0.0	0.001	100	0.80	100.0	0.0	0.001
50	0.90	50.0	0.0	0.001	100	0.90	100.0	0.0	0.001

Table 1: Full report of the results obtained by the asymmetric formulation on the benchmark set of instances **random**. Columns Sol, Gap, and Time represent the best solution, the optimality gap, and the elapsed time as reported by the solver, respectively. The times are reported in seconds. If the formulation exceeds the time limit a *TL* appears in that entry in the table.

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