

INTERVAL SOLUTION FOR GENERALIZED LINEAR-FRACTIONAL OPTIMIZATION PROBLEMS WITH INTERVAL PARAMETERS

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ABSTRACT

Generalized linear-fractional optimization refers to the maximization (minimization) of the minimum (maximum) value over a finite set of linear-fractional objective functions subject to linear constraints. We address generalized linear-fractional problems with interval parameters both in the objective functions and the constraints. By introducing consistency checks for systems of linear inequalities, we show that the endpoints of the interval that contains the optimal value of an arbitrary instance of the interval problem are the optimal values of exactly two deterministic generalized linear-fractional problems. Illustrative applications demonstrate the characteristics and the numerical performance of the proposed solution method.

KEYWORDS. Linear-Fractional Programming. Optimization under Uncertainty. Optimization Algorithms.

Paper topics: MP - Mathematical Programming

1. Introduction

The linear-fractional optimization problem consists in the minimization or maximization of a ratio of affine functions subject to linear constraints. The linear-fractional problem arises from the necessity of optimizing the efficiency of some activity, as maximizing profit per unit of production cost or minimizing production cost per unit produced by some firm, among other formulations [Bajalinov, 2003]. Linear-fractional optimization methods are especially relevant while managing scarce resources, as for example, natural resources. The solution of the linear-fractional problem reduces to the solution of an equivalent linear problem by means of a variables transformation [Charnes and Cooper, 1962], or to the solution of a sequence of parameterized linear problems, which is the central aspect of the so-called Dinkelbach's algorithm [Dinkelbach, 1967]. See [Bajalinov, 2003] for a comprehensive exposition of concepts and methods of linear-fractional optimization.

The linear-fractional problem has been generalized in the sense of replacing the maximization of a single linear-fractional objective function with the maximization of the minimum (maximin) or with the maximization of the sum (maxisum) of two or more linear-fractional objective functions [Schaible, 1995]. Generalized linear-fractional problems in the maximin (or minimax) sense are considered in this paper. The solution of linear-fractional maximin problems can be efficiently accomplished by extensions of the Dinkelbach's algorithm [Crouzeix et al., 1985].

Concepts and methods of fractional optimization having the linear-fractional problem as a special case are discussed in [Schaible, 1995]. We observe that almost all the literature dedicated to this important branch of the mathematical optimization assumes that the problem is deterministically formulated, although problems derived from real applications may exhibit uncertainties in parameters of the objective functions or the constraints. Interval uncertainties, when parameters are described by real intervals, and fuzzy uncertainties, when the parameters are represented by fuzzy numbers, are the most frequent representations of uncertain parameters in the literature.

The generalized linear-fractional problem with interval parameters was introduced in [Hladik, 2010]. Its resolution consists in finding the lower and upper endpoints of the interval of minimum width that limits the optimal value of an arbitrary deterministic instance of the problem. [Hladik, 2010] proposes methods for obtaining these endpoints for the special case of ratios of linear functions, as well as for the converse problem of guaranteeing that the optimal value remains within prescribed endpoints. Other effective methods are developed based on characterizations of bounds for the linear-fractional functions over the feasible region. One of these characterizations, presented in [Effati and Pakdaman, 2012], is employed in [Nayak and Ojha, 2015] for generating Pareto-optimal solutions of multiobjective linear-fractional optimization problems. More recently, [Abad et al., 2020] proposes two methods for the solution of the interval, single-objective, linear-fractional problem, which consist in the computation of bounds for its optimal value.

In this paper we propose a solution method for generalized linear-fractional problems with interval parameters. Our contributions are the following: (i) We derive endpoints for the optimal values of the interval problem for the general case of ratios of affine functions. In our derivation, interval objectives and constraints, with the exception of interval denominators, which ultimately will condition the endpoints, are progressively replaced by suitable noninterval representations; (ii) By incorporating at most two consistency checks for systems of linear inequalities, the endpoints of the interval problem are obtained as the optimal values of exactly two deterministic generalized linear-fractional problems, for which the Dinkelbach's algorithm is usually very efficient; (iii) The effectiveness of the proposed solution method is demonstrated by its application to selected problems in Economics and Chemical Engineering.

The rest of the article is organized in four sections. In Section 2 we present concepts and results of linear-fractional optimization, the formulation of the generalized linear-fractional problem, and its standard solver, the Dinkelbach's algorithm. In Section 3 we review basic operations of Interval Arithmetic, consider the generalized linear-fractional problem with interval parameters, derive the endpoints for its optimal values, and compare their computation with an alternative procedure available in the literature. In Section 4 we develop applications of the proposed method to selected problems in Economics and Chemical Engineering. Finally, in Section 5 we present our conclusions and topics for future research.

2. Generalized Linear-Fractional Optimization

We consider a set of m linear-fractional functions q_i defined as

$$q_i(x) = \frac{p_i(x)}{d_i(x)} = \frac{\sum_{j=1}^n p_{ij}x_j + p_{i0}}{\sum_{j=1}^n d_{ij}x_j + d_{i0}}, \quad i = 1, 2, \dots, m,$$

and a polyhedral feasible region $\mathcal{X} \subset \mathbb{R}^n$ represented in the form

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n a_{rj}x_j \leq b_r, \quad r = 1, 2, \dots, s, \quad x_j \geq 0, \quad j = 1, 2, \dots, n \right\}. \quad (1)$$

We initially assume that the parameters $p_{i0}, p_{ij}, d_{i0}, d_{ij}, a_{rj}$ and b_r are precisely known, and that $d_i(x) \neq 0$ for all $x \in \mathcal{X}$ and all $i = 1, 2, \dots, m$. Then, without any loss of generality [Bajalinov, 2003], we can consider that $d_i(x) > 0$ for all $x \in \mathcal{X}$ and all $i = 1, 2, \dots, m$. The mono-objective linear-fractional problem is

$$\underset{x \in \mathcal{X}}{\text{maximize}} \quad q_1(x). \quad (2)$$

See [Bajalinov, 2003] for a comprehensive presentation of concepts and methods of mono-objective linear-fractional optimization. In this paper we address multiobjective linear-fractional problems,

$$\underset{x \in \mathcal{X}}{\text{maximize}} \quad \{q_1(x), q_2(x), \dots, q_m(x)\}, \quad m \geq 2, \quad (3)$$

in the maximin (conservative) sense:

$$\underset{x \in \mathcal{X}}{\text{maximize}} \quad \min_{1 \leq i \leq m} \{q_i(x)\}. \quad (4)$$

We assume that $\mathcal{X}$ is bounded and introduce the linear optimization problem parameterized by $\lambda \in \mathbb{R}$

$$\underset{x \in \mathcal{X}}{\text{maximize}} \quad \min_{1 \leq i \leq m} \{p_i(x) - \lambda d_i(x)\}, \quad (5)$$

whose optimal value function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is

$$\phi(\lambda) = \max_{x \in \mathcal{X}} \min_{1 \leq i \leq m} \{p_i(x) - \lambda d_i(x)\}. \quad (6)$$

The function ϕ is continuous and strictly decreasing. The Dinkelbach's algorithm finds (λ^*, x^*) such that $\phi(\lambda^*) = 0$, and then it is shown that x^* is an optimal solution and $f^* = \lambda^*$ is the optimal value of the maximin problem (4) [Schaible, 1995].

Generalized Dinkelbach's Algorithm

Step 1. (Initialization) Determine $x^0 \in \mathcal{X}$,

$$\lambda^0 = \min_{1 \leq i \leq m} \{q_i(x^0)\}$$

and set $k = 1$;

Step 2. Solve the parametric linear problem (5). Let x^k be the optimal solution obtained;

Step 3. If $\phi(\lambda^k) = 0$, then stop: x^k is an optimal solution of the maximin problem (4) and $\lambda^k = \lambda^* = f^*$. Otherwise, let

$$\lambda^{k+1} = \min_{1 \leq i \leq m} \{q_i(x^k)\},$$

replace k with $k + 1$ and go back to Step 2.

The sequence (λ^k) generated by the Dinkelbach's algorithm is increasing and converges linearly to λ^* . Any convergent subsequence of (x^k) converges to an optimal solution of problem (4). The convergence properties of the algorithm for $m = 1$ are superior [Schaible, 1995].

3. Generalized Linear-Fractional Optimization with Interval Parameters

In this section we consider generalized linear-fractional optimization problems with interval parameters. We assume as known the scalars $p_{ij}^L \leq p_{ij}^U$, $p_0^L \leq p_0^U$, $d_{ij}^L \leq d_{ij}^U$, $d_0^L \leq d_0^U$, $a_{rj}^L \leq a_{rj}^U$ and $b_r^L \leq b_r^U$, the endpoints of closed intervals that represent the uncertainties about the real parameters of the linear-fractional functions and functional constraints:

$$p_{ij} \in p_{ij}^I = [p_{ij}^L, p_{ij}^U], \quad p_{i0} \in p_{i0}^I = [p_{i0}^L, p_{i0}^U], \quad d_{ij} \in d_{ij}^I = [d_{ij}^L, d_{ij}^U], \quad d_{0i} \in d_{0i}^I = [d_{0i}^L, d_{0i}^U], \\ a_{rj} \in a_{rj}^I = [a_{rj}^L, a_{rj}^U], \quad b_r \in b_r^I = [b_r^L, b_r^U],$$

for all $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ and $r = 1, 2, \dots, s$. The superindices I , L e U denote interval, lower endpoint and upper endpoint of an interval, respectively. Concepts and methods of Interval Analysis are presented in [Alefeld and Hertzberger, 1983]. In this article, only a limited number of concepts and arithmetic operations involving intervals are employed. Let $a^I = [a^L, a^U]$, $a^L \leq a^U$, and $b^I = [b^L, b^U]$, $b^L \leq b^U$, be closed intervals. We say that an interval a^I is *punctual* if $a^L = a^U$. The *width* of a^I is the real quantity $a^U - a^L$. Basic arithmetic operations with intervals are presented below.

- (i) Sum: $a^I + b^I = [a^L + b^L, a^U + b^U]$;
- (ii) Difference: $a^I - b^I = [a^L - b^U, b^U - a^L]$;
- (iii) Product by Scalar: $\alpha a^I = [\alpha a^L, \alpha a^U]$ if $\alpha \geq 0$; $\alpha a^I = [\alpha a^U, \alpha a^L]$ if $\alpha < 0$;
- (iv) Product: $a^I b^I = [\min \mathcal{P}, \max \mathcal{P}]$, $\mathcal{P} = \{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$;
- (v) Division: If $0 \notin b^I$, then $a^I / b^I = [\min \mathcal{D}, \max \mathcal{D}]$, $\mathcal{D} = \{a^L / b^L, a^L / b^U, a^U / b^L, a^U / b^U\}$.

The interval extension of the feasible region (1) is defined as

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n a_{rj}^I x_j \leq b_r^I, \quad r = 1, 2, \dots, s, \quad x_j \geq 0, \quad j = 1, 2, \dots, n \right\}, \quad (7)$$

and comprises all the nonnegative vectors x for which

$$\sum_{j=1}^n a_{rj}x_j \leq b_r$$

holds for all $a_{rj} \in a_{rj}^I$ and $b_r \in b_r^I$, $r = 1, 2, \dots, s$, $j = 1, 2, \dots, n$. For convenience, we also introduce the following sets:

$$\mathcal{X}^U = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n a_{rj}^L x_j \leq b_r^U, \quad r = 1, 2, \dots, s, \quad x_j \geq 0, \quad j = 1, 2, \dots, n \right\}, \quad (8)$$

$$\mathcal{X}^L = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n a_{rj}^U x_j \leq b_r^L, \quad r = 1, 2, \dots, s, \quad x_j \geq 0, \quad j = 1, 2, \dots, n \right\}. \quad (9)$$

Proposition 3.1 $\mathcal{X}^L \subset \mathcal{X} \subset \mathcal{X}^U$

Proof. Suppose that $x \in \mathcal{X}$. Then for $r = 1, 2, \dots, s$ and $j = 1, 2, \dots, n$,

$$\sum_{j=1}^n a_{rj}x_j \leq b_r, \quad x_j \geq 0,$$

for all $a_{rj} \in a_{rj}^I$ and $b_r \in b_r^I$. Since $a_{rj} \geq a_{rj}^L$ and $b_r \leq b_r^U$, it follows that $x \in \mathcal{X}^U$. Now suppose that $x \in \mathcal{X}^L$. Then

$$\sum_{j=1}^n a_{rj}^U x_j \leq b_r^L, \quad x_j \geq 0,$$

and given that $a_{rj} \leq a_{rj}^U$ and $b_r \geq b_r^L$ for all $a_{rj} \in a_{rj}^I$ and $b_r \in b_r^I$, we conclude that $x \in \mathcal{X}$. $\square$

Since $x_j \geq 0$, $j = 1, 2, \dots, n$, the following interval inclusions hold for the numerators and denominators of the linear-fractional functions: for $i = 1, 2, \dots, m$ and all the nonnegative vectors x ,

$$p_i(x) \in p_i^I(x) = [p_i^L(x), p_i^U(x)] = \left[\sum_{j=1}^n p_{ij}^L x_j + p_{i0}^L, \sum_{j=1}^n p_{ij}^U x_j + p_{i0}^U \right],$$

$$d_i(x) \in d_i^I(x) = [d_i^L(x), d_i^U(x)] = \left[\sum_{j=1}^n d_{ij}^L x_j + d_{i0}^L, \sum_{j=1}^n d_{ij}^U x_j + d_{i0}^U \right].$$

We assume that $\mathcal{X}^U$ is bounded and that $d_i^L(x) > 0$ ($i = 1, 2, \dots, m$) for all $x \in \mathcal{X}^U$. The positivity of denominators is a working hypothesis often ensured in many applications of linear-fractional optimization. A less restrictive hypothesis – nonnegativity rather than positivity – is adopted in [Hladik, 2010]. Our assumption allows us to guarantee that, for an arbitrary instance of interval parameters,

$$\frac{p_i^L(x)}{d_i(x)} \leq \frac{p_i(x)}{d_i(x)} \leq \frac{p_i^U(x)}{d_i(x)}, \quad i = 1, 2, \dots, m,$$

for all $x \in \mathcal{X}$. Then, from Proposition 3.1, it follows that

$$\max_{x \in \mathcal{X}} \min_{1 \leq i \leq m} \left\{ \frac{p_i(x)}{d_i(x)} \right\} \geq \max_{x \in \mathcal{X}^L} \min_{1 \leq i \leq m} \left\{ \frac{p_i^L(x)}{d_i(x)} \right\}, \quad (10)$$

$$\max_{x \in \mathcal{X}} \min_{1 \leq i \leq m} \left\{ \frac{p_i(x)}{d_i(x)} \right\} \leq \max_{x \in \mathcal{X}^U} \min_{1 \leq i \leq m} \left\{ \frac{p_i^U(x)}{d_i(x)} \right\}. \quad (11)$$

Solving the generalized linear-fractional maximin problem with interval parameters consists in finding the interval $[f^L, f^U]$ of minimum width which contains the optimal value of the problem for any possible instance of its interval parameters. The lower (f^L) and the upper (f^U) endpoints of this interval can be found by determining $d_i(x) \in d_i^I(x)$, $i = 1, 2, \dots, m$, (that is, $d_{ij} \in d_{ij}^I$, $d_{0i} \in d_{0i}^I$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$) such that the right-hand sides of (10) and (11) attain their minimum and maximum, respectively. For this purpose, consider the following optimal values:

$$f_1 := \max_{x \in \mathcal{X}^L} \min_{1 \leq i \leq m} \left\{ \frac{p_i^L(x)}{d_i^L(x)} \right\}, \quad f_2 := \max_{x \in \mathcal{X}^L} \min_{1 \leq i \leq m} \left\{ \frac{p_i^L(x)}{d_i^U(x)} \right\}, \quad (12)$$

$$f_3 := \max_{x \in \mathcal{X}^U} \min_{1 \leq i \leq m} \left\{ \frac{p_i^U(x)}{d_i^U(x)} \right\}, \quad f_4 := \max_{x \in \mathcal{X}^U} \min_{1 \leq i \leq m} \left\{ \frac{p_i^U(x)}{d_i^L(x)} \right\}. \quad (13)$$

It is worth noting that, as $d_i(x)$ is assumed to be positive for every instance $d_i(x) \in d_i^I(x)$, the signs of f_1 , f_2 , f_3 and f_4 solely depend on those of $p_i^L(x)$ and $p_i^U(x)$ ($i = 1, 2, \dots, m$).

Theorem 3.1 *The endpoints of the generalized linear-fractional maximin problem with interval parameters are*

$$f^L = \begin{cases} f_1, & \text{if } f_1 < 0; \\ f_2, & \text{otherwise.} \end{cases} \quad \text{and} \quad f^U = \begin{cases} f_3, & \text{if } f_3 < 0; \\ f_4, & \text{otherwise.} \end{cases}$$

Proof. (Lower endpoint) Suppose that f^L is negative, indicating that the right-hand side of (10) is negative for at least one instance $d_i(x) \in d_i^I(x)$ ($i = 1, 2, \dots, m$). Then for this instance and all $x \in \mathcal{X}^L$, there exists at least one index k for which $p_k^L(x)$ is negative. This means that the system of linear inequalities

$$p_i^L(x) \geq 0, \quad i = 1, 2, \dots, m, \quad x \in \mathcal{X}^L, \quad (14)$$

is inconsistent, and the right-hand side of (10) is negative for every instance $d_i(x) \in d_i^I(x)$ ($i = 1, 2, \dots, m$). In this case, the lower endpoint is established by the instance $d_i^L(x)$ ($i = 1, 2, \dots, m$), because, for all $x \in \mathcal{X}^L$,

$$\min_{1 \leq i \leq m} \left\{ \frac{p_i^L(x)}{d_i^L(x)} \right\} \leq \min_{1 \leq i \leq m} \left\{ \frac{p_i^L(x)}{d_i(x)} \right\} < 0,$$

implying that $f^L = f_1$ ($f_1 < 0$). If (14) is consistent ($f_1 \geq 0$), the right-hand side of (10) is nonnegative for every instance $d_i(x) \in d_i^I(x)$ ($i = 1, 2, \dots, m$). The lower endpoint is established by the instance $d_i^U(x)$ ($i = 1, 2, \dots, m$), because, for all x satisfying (14),

$$0 \leq \min_{1 \leq i \leq m} \left\{ \frac{p_i^L(x)}{d_i^U(x)} \right\} \leq \min_{1 \leq i \leq m} \left\{ \frac{p_i^L(x)}{d_i(x)} \right\}.$$

We conclude that $f^L = f_1$ if $f_1 < 0$ and $f^L = f_2$, otherwise. (Upper endpoint) Suppose that f^U is negative. Then the right-hand side of (11) is negative for every instance $d_i(x) \in d_i^I(x)$ ($i = 1, 2, \dots, m$), the system of linear inequalities

$$p_i^U(x) \geq 0, \quad i = 1, 2, \dots, m, \quad x \in \mathcal{X}^U, \quad (15)$$

is inconsistent and, given that

$$0 > \min_{1 \leq i \leq m} \left\{ \frac{p_i^U(x)}{d_i^U(x)} \right\} \geq \min_{1 \leq i \leq m} \left\{ \frac{p_i^U(x)}{d_i(x)} \right\}$$

for all $x \in \mathcal{X}^U$, we have $f^U = f_3$ ($f_3 < 0$). If (15) is consistent ($f_3 \geq 0$), the right-hand side of (10) is nonnegative for every instance $d_i(x) \in d_i^I(x)$ ($i = 1, 2, \dots, m$). The upper endpoint is established by the instance $d_i^L(x)$ ($i = 1, 2, \dots, m$), because, for all x satisfying (15),

$$\min_{1 \leq i \leq m} \left\{ \frac{p_i^U(x)}{d_i^L(x)} \right\} \geq \min_{1 \leq i \leq m} \left\{ \frac{p_i^U(x)}{d_i(x)} \right\} \geq 0.$$

We conclude that $f^U = f_3$ if $f_3 < 0$ and $f^U = f_4$, otherwise. $\square$

As $p_i^U(x) \geq p_i^L(x)$ ($i = 1, 2, \dots, m$) for all $x \in \mathcal{X}^L \subset \mathcal{X}^U$, the system of linear inequalities (15) is consistent if (14) is consistent. In this case, $f^L = f_2$ and $f^U = f_4$. If (14) is inconsistent, then $f^L = f_1$, and the consistency check of (15) leads to $f^U = f_3$ or $f^U = f_4$. Thus, incorporating at most two consistency checks to the determination of f^L and f^U , exactly two standard generalized linear-fractional problems, for which the Dinkelbach's algorithm is usually very efficient, need to be solved. The consistency of systems of linear inequalities can be verified, for example, using the Phase I procedure of the Simplex Method [Bazaraa et al., 1990].

The structure of Theorem 3.1 resembles that of Theorem 1 of Hladik [2010], in which the corresponding computation of f_1 , using our notations, would be

$$\begin{aligned}
 & \underset{x, \lambda}{\text{maximize}} && \lambda \\
 & \text{subject to} && p_i^L(x) \geq \lambda d_i^L(x), \quad i = 1, 2, \dots, m, \\
 & && x \in \mathcal{X}^L, \lambda \leq 0,
 \end{aligned} \quad (16)$$

where $p_i^L(x)$ and $d_i^L(x)$ are linear and $d_i^L(x) \geq 0$ ($i = 1, 2, \dots, m$) for all $x \in \mathcal{X}$. Problem (16) can not be directly solved using linear optimization-based algorithms as the Dinkelbach's algorithm (Section 2). The endpoints f^L and f^U are determined by solving up to four problems of the form (16) through interior point methods. Differently from Hladik [2010], we consider ratios of affine functions, exactly two standard generalized linear-fractional problems need to be solved, and the use of the Dinkelbach's algorithm is ensured. The positivity of $d_i(x) \in d_i^I(x)$ ($i = 1, 2, \dots, m$) greatly simplifies the derivation, characterization and computation of f^L and f^U .

4. Illustrative Applications

In this section we discuss applications of the proposed method to optimization problems in Economics and Chemical Engineering. All numerical computations were performed in GNU Octave [Eaton et al., 2014] with GLPK [GLPK, 2014] as linear optimization solver. The convergence criterion for the Dinkelbach's Algorithm (Section 2) was $|\phi(\lambda^k)| < 10^{-6}$.

4.1. Application in Economics

This problem refers to a von Neumann-Leontief economic growth model, formulated and solved in [Li, 2010] using a formalism from Economics, and solved in [Hladik, 2010] using the generalized linear-fractional optimization methods proposed. The problem has six decision variables, which represent activity levels of six economic sectors, and four linear-fractional functions, which represent the production of four goods. The linear-fractional functions are

$$\begin{aligned} q_1(x) &= \frac{x_1}{0.28x_1 + 0.50x_2 + 0.53x_3}, & q_2(x) &= \frac{x_2 + x_5}{0.84x_1 + 0.77x_6}, \\ q_3(x) &= \frac{0.25x_2 + x_3 + x_4 + 0.25x_5}{0.49x_2 + 0.45x_3 + 0.50x_4 + 0.48x_5}, & q_4(x) &= \frac{x_6}{0.51x_4 + 0.57x_5 + 0.29x_6}, \end{aligned}$$

and the feasible region is

$$\mathcal{X} = \{x \in \mathbb{R}^6 : x_1 + x_2 + \dots + x_6 = 1, x_j > 0, j = 1, 2, \dots, 6\}. \quad (17)$$

The optimal value of the maximin problem provides the growth factor of the economy. Applying the Dinkelbach's algorithm with the above nominal data, we found (with $x_j > 10^{-16}$, $j = 1, 2, \dots, n$, after 12 iterations) the same normalized optimal vector obtained in [Li, 2010], $x^* = [0.2725 \ 0.3639 \ 0.0000 \ 0.2038 \ 0.0000 \ 0.1568]^T$. The optimal value of the problem, $f^* = 1.0492$, indicates that the economy is slightly expanding.

The interval version of the problem was generated as in [Hladik, 2010], replacing the nominal value θ of each positive parameter of q_i ($i = 1, 2, 3, 4$) with an interval $[(1 - \delta)\theta, (1 + \delta)\theta]$, where $\delta > 0$ is some (usually small) quantity that reflects uncertainty about the real value of θ . As for $\delta < 1$ all lower endpoints of all intervals are positive, the system of linear inequalities (14) is consistent (Theorem 3.1), and therefore $f^L = f_2$ and $f^U = f_4$. Using the Dinkelbach's algorithm for computing these endpoints, we searched for the largest δ assuring that $f^* \geq 1$, and obtained $\delta^* = 0.024$ and $1 \leq f^* \leq 1.1$ (that is, $f^L = 1$ and $f^U = 1.1$). One of the methods proposed in [Hladik, 2010] aimed at guaranteeing that $1 \leq f^* \leq 1.2$, and also found $\delta^* = 0.024$. For each δ , the proposed method solves two standard generalized problems, while the method proposed in [Hladik, 2010] solves three non-standard generalized problems. $\square$

4.2. Application in Chemical Engineering

Thermal Cracking is the chemical process by which large organic molecules as hydrocarbons present in crude oil have their bonds broken to generate simpler molecules, as lighter hydrocarbons [Sadrameli, 2015]. The optimal operation of a thermal cracker is formulated in [Edgar et al., 2001] as a linear optimization problem. The process involves the conversion of four reactants (feed) – propane, ethane, gas oil and debutanized natural gasoline (DNG) – into eight products – ethylene, propylene, butadiene, gasoline, methane, ethane (recycle), propane (recycle) and fuel oil. See [Edgar et al., 2001] for a detailed discussion about such conversion process. The following seven variables represent flow rates to and from the reactor (in lb/h): x_1 , ethane; x_2 , propane; x_3 , gas oil; x_4 , GND; x_5 , ethane (recycle); x_6 , propane (recycle); x_7 , fuel added. The objective is to maximize the operational profit of the reactor, expressed as

$$p(x) = r(x) - (c(x) + c_0),$$

where r , c e c_0 denote, respectively, the products revenue (in cents per hour), the reactants cost (cent/h) and a constant energy cost (cent/h) to compensate the heat loss in the reactor. The two linear formulations presented in [Edgar et al., 2001] will be explored. In the first, the reactor operational constraints are

1. Cracker capacity (200000 lb/h):

$$1.1x_1 + 0.9x_2 + 0.9x_3 + 1.0x_4 + 1.1x_5 + 0.95x_6 \leq 200000;$$

2. Ethylene processing limitation (50000 lb/h):

$$0.5x_1 + 0.35x_2 + 0.25x_3 + 0.25x_4 + 0.5x_5 + 0.35x_6 = 50000;$$

3. Propylene processing limitation (20000 lb/h):

$$0.01x_1 + 0.15x_2 + 0.15x_3 + 0.18x_4 + 0.01x_5 + 0.15x_6 = 20000;$$

4. Ethane recycle: $0.4x_1 + 0.06x_2 + 0.04x_3 + 0.05x_4 - 0.6x_5 + 0.06x_6 = 0$;

5. Propene recycle: $0.1x_2 + 0.01x_3 + 0.01x_4 - 0.9x_6 = 0$;

6. Heat constraint:

$$-6857.6x_1 + 364x_2 + 2.032x_3 - 114x_4 - 6857.6x_5 + 364x_6 + 21520x_7 = 20000000;$$

7. Nonnegativity constraints: $x_i \geq 0$, $i = 1, 2, \dots, 7$.

As in [Edgar et al., 2001], we assume that $c_0 = 115.777, 6$ (cent/h), that the products revenue is given by

$$r(x) = 9.39x_1 + 9.51x_2 + 9.17x_3 + 11.23x_4 + 9.39x_5 + 9.51x_6, \quad (18)$$

but we introduce three different scenarios for the reactants costs:

$$c_1(x) = 6.55x_1 + 9.73x_2 + 12.50x_3 + 10.14x_4,$$

$$c_2(x) = 13.10x_1 + 7.30x_2 + 8.40x_3 + 12.60x_4,$$

$$c_3(x) = 12.90x_1 + 8.30x_2 + 11.50x_3 + 10.10x_4.$$

The unit costs of c_1 are those adopted in [Edgar et al., 2001]. The unit costs of c_2 and c_3 were chosen to generate relatively distinct production plans. The proposed generalized linear-fractional problem consists in optimizing the reactor operation in the maximin sense (4), with $p_i(x) = r(x) - c_i(x) - c_0$ and $d_i(x) = c_i(x) + c_0$ ($i = 1, 2, 3$). We observe that $d_i(x) > 0$, $i = 1, 2, 3$, over the problem feasible region ($\mathcal{X} \subset \mathbb{R}^7$). Table 1 presents the optimal solutions of the individual linear-fractional problems, that is, when each fraction is optimized individually subject to all the constraints. Table 2 presents the optimal solution of the maximin problem (4).

An additional investigation shows that the objective value (profit) of the linear-fractional problem is $-0,0632$, $-0,0016$ and $0,0846$ when the optimal solutions for q_1 , q_2 and q_3 are implemented under the scenarios 2, 1 and 1, respectively. On the other hand, the optimal solution of the maximin problem assures a minimum profit of 13 cents for each dollar spent (13 cents if scenarios 1 or 2 occur; 18 cents if scenario 3 occurs). An interval version of the problem is obtained assuming that the unit cost c_{ij} of reactant j in the cost scenario i is replaced by an interval of the form

$$c_{ij}^I = [(1 - \delta)c_{ij}^0, (1 + \delta)c_{ij}^0],$$

Value	q_1	q_2	q_3
x_1	21768.1	0	0
x_2	0	95834.1	101902.8
x_3	0	25738.8	0
x_4	107594.5	0	15905.7
x_5	23597.8	12392.8	12665.7
x_6	1195.5	10934.2	11499.3
x_7	21090.2	642.2	3893.6
f^*	0.2196	0.3259	0.2254

Table 1: Optimal solutions for the three scenarios.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	f^*
7370.7	67398.6	0	46951.5	16367.3	8010.4	9716.4	0.1322

Table 2: Optimal solution of the maximin problem.

where c_{ij}^0 is the nominal cost and $\delta > 0$ is a scalar that reflects the uncertainty about the true value of c_{ij} . The unit profits (computed through the arithmetic operations presented in Section 3) are the intervals given by

$$p_{ij}^I = r_{ij}^I - d_{ij}^I, \quad d_{ij}^I = c_{ij}^I + c_0^I, \quad i = 1, 2, 3, \quad j = 1, 2, \dots, 7, \quad (19)$$

with $c_{ij}^I = [0, 0]$ for $i = 1, 2, 3$ and $j = 5, 6, 7$. The unit revenues and the constant c_0 are assumed precisely known, which implies that r_{ij}^I ($i = 1, 2, 3, j = 1, 2, \dots, 7$) and c_0^I are punctual intervals. Table 3 presents the endpoints on the optimal solutions of the generalized linear-fractional problem with interval costs for different values of δ .

δ (in %)	0.1	1	5	10	15	20	25
f^L	0.1312	0.1220	0.0829	0.0377	-0.0050	-0.0610	-0.1235
f^U	0.1333	0.1426	0.1862	0.2456	0.3113	0.3843	0.4659

Table 3: Endpoints of the interval maximin problem for different values of δ .

In Table 3, each interval $[f^L, f^U]$ contains the optimal value of the deterministic maximin problem ($f^* = 0.1322$) for all values δ tested, and we observe a relatively slow increase of the width of $[f^L, f^U]$ as δ increases from 0.1% to 25%. The method proposed would guarantee an approximate profit of 4 cents per dollar invested when $\delta = 10\%$.

The second problem formulation discussed in [Edgar et al., 2001] will be used for illustrating the performance of the proposed method when, in addition to uncertainties in parameters of the fractional functions, there are also uncertainties in parameters of the constraints. The second (ethylene processing limitation) and the third (propylene processing limitation) equality constraints are replaced with inequality constraints. Using the remaining equality constraints, we eliminate variables x_5 , x_6 and x_7 and the following expressions hold:

1. Revenue: $r(x) = 15.65x_1 + 11.61x_2 + 9.91x_3 + 12.13x_4$;
2. Profit: $p_i(x) = r(x) - c_i(x) - c_0, \quad i = 1, 2, 3$;

3. Cracker capacity: $1.83x_1 + 1.12x_2 + 0.98x_3 + 1.10x_4 \leq 200000$;
4. Ethylene processing limitation: $0.83x_1 + 0.44x_2 + 0.29x_3 + 0.29x_4 \leq 50000$;
5. Propylene processing limitation: $0.02x_1 + 0.17x_2 + 0.15x_3 + 0.18x_4 \leq 20000$;
6. Nonnegativity constraints: $x_j \geq 0, j = 1, 2, 3, 4$.

The same strategy adopted for generating interval fractional functions is used for generating interval constraints: for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$,

$$a_{ij}^I = [(1 - \delta_a)a_{ij}^0, (1 + \delta_a)a_{ij}^0], \quad b_i^I = [(1 - \delta_b)b_i^0, (1 + \delta_b)b_i^0],$$

where a_{ij}^0 and b_i^0 are the nominal (central) values of a_{ij}^I and b_i^I , and $\delta_a > 0$ and $\delta_b > 0$ are scalars that introduce uncertainties about the true values of a_{ij} and b_i , respectively. For simplicity, we have assumed that the parameters a_{ij} and b_i are subject to the same sources of uncertainty. The optimal objective function value of the deterministic maximin problem ($\delta = \delta_a = \delta_b = 0$) is $f^* = 0.1958$. Table 4 presents the bounds of the interval problem for $\delta = 5\%$ and selected values of δ_a and δ_b .

(δ_a, δ_b) (in %)	f^L	f^U
(1, 1)	0.1425	0.2541
(1, 5)	0.1371	0.2597
(1, 10)	0.1299	0.2663
(1, 20)	0.1129	0.2779
(5, 5)	0.1319	0.2656
(5, 10)	0.1244	0.2719
(5, 20)	0.1070	0.2832
(10, 10)	0.1177	0.2790
(10, 20)	0.0997	0.2898
(20, 20)	0.0854	0.3032

Table 4: Endpoints of the interval maximin problem for $\delta = 5\%$ and different values of δ_a and δ_b .

In Table 4, each interval $[f^L, f^U]$ contains the optimal value of the deterministic maximin problem and, as δ_a and δ_b increase, a relatively slow increase in the width of $[f^L, f^U]$ is observed. The decrease of f^L is slightly faster than the increase of f^U , and changes in δ_b has a more pronounced impact on f^L and f^U .

5. Acknowledgement

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 - Process Number 88887.640902/2021-00 of the PROEX Program.

6. Conclusions

In this paper we proposed an interval solution for generalized linear-fractional problems with interval parameters. The solution method is implemented through the linear optimization-based Dinkelbach's Algorithm. Appending the information about the consistency of at most two systems of linear inequalities, the interval solution of the problem is obtained by solving exactly two deterministic generalized linear-fractional problems. Two illustrative applications selected from the literature showed that the method is able to confine optimal values within intervals of relatively small widths. The authors are currently investigating possible extensions of the proposed method and other applications of generalized linear-fractional optimization.

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