

A new feasibility quality measure for optimization under uncertainty

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ABSTRACT

This research introduces a new metric that measures the quality of solutions arising from uncertain optimization problems called hardness. The measure considers the intensity with which a specific solution of an optimization problem under uncertainty can violate the set of constraints. This is done in a comparative way, which is not found in other quality measures present in the literature. Another attractive feature is the possibility of the decision maker establish the importance of each constraint during the metric evaluation process, which gives more flexibility to a posterior analysis. The hardness measure presented better performance than the other metric presented in the literature, being able to describe the difference between solutions with intermediate uncertainty protection in a detailed way. It is concluded that the hardness quality measure, when carrying out a relative feasibility comparison weighted by the decision maker's preferences, is the best choice to identify the most protected solution.

KEYWORDS. Robustness and sensitivity analysis. Optimization Under Uncertainty. Quality Measure.

Robustness and sensitivity analysis, Optimization Under Uncertainty, Quality Measure, Feasibility Measure, Robustness Measure

1. Optimization under uncertainty

An optimization problem under uncertainty (OPUU) is a family of optimization problems that share the same structure but with different parameter values. Those parameters are *uncertain* and belongs to a *uncertain set* [Ben-Tal et al., 2009].

Definition: OPUU Given an objective (minimization/maximization), a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, a constraint function $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$ and a non empty set $\mathcal{V} \subset \mathbb{R}^q$, an OPUU associated with it can be defined as:

$$\{\min f(\vec{x}) : g(\vec{x}, \vec{v}) \leq 0\}_{\vec{v} \in \mathcal{V}}, \quad (1)$$

where $\mathcal{V}$ is called *uncertain set* and $\vec{v}$ is called *uncertain parameter*.

Consider the situation where the uncertain set is described by boundary information for each uncertain parameter. One could chose to find the robust solution for the OPUU [Bertsimas et al., 2011; Bertsimas e Sim, 2004; Greenberg e Morrison, 2008], that is, find the best solution that will be feasible for all optimization problems that belong to the OPUU family. This research will present two quality measures for comparing OPUU solutions: the probability of constraint violation measure and a new one, the hardness measure. In order to demonstrate the efficacy of these metrics, a OPUU will be solved by a robust optimization approach for five different uncertainty protection levels.

The main contribution of this research is the definition and verification of the efficacy measure called hardness. Our new measure is useful because it indicates how solutions compare (better/worse) to all possible solutions, also considering the decision maker's preferences. The constraint set violation probability is not a comparative measure and it is not sensitive to preferences. The hardness measure can compare solutions given by any method for a OPUU that has at least one possible solution. It has 1 as a upper bound, meaning that a solution is robust. The closer the measure gets to 0, the less the solution is robust.

2. Comparison measures for optimization under uncertainty

2.1. Constraint Set Violation probability

The first measure is called *constraint set violation probability* [Calafiore e Campi, 2005; Campi e Garatti, 2008]. Given the OPUU (1) with a constraint function $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$ and a uncertain set $\mathcal{V} \subset \mathbb{R}^q$, the constraint set violation probability measure $\zeta : \mathbb{R}^n \rightarrow [0, 1]$ is defined as

$$\zeta(\vec{x}) = Pr \left(\sum_{j=1}^m (g(\vec{x}, \vec{v}))_j^+ > 0 : \vec{v} \in \mathcal{V} \right) \quad (2)$$

where the superscript plus sign is the function $(\cdot)^+ : \mathbb{R} \rightarrow [0, \infty)$ defined as $(x)^+ = 0$ if $x < 0$ and $(x)^+ = x$ otherwise.

The constraint set violation probability arise in the context of chance constraint optimization [Charnes e Cooper, 1959; Samsatli et al., 1998]. This approach reduces the OPUU to a deterministic optimization problem, where the violation probability is constrained by an upper bound. This measure lack of sensitivity and can give an impression to the decision maker that there is no difference between solutions from the robust solution to any other with a lower uncertainty protection.

2.2. Hardness

The second measure is a new measure called the *hardness* measure. It is a feasibility measure that arises in the context of robust optimization and has the ability to classify the uncertainty protection level in a comparative way. This feature makes hardness an useful indicator for a decision making process that involves the solution of an OPUU.

Consider the OPUU defined by (1). Let $\beta : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function, called *positive brittleness* (PB), defined as

$$\beta^+(\vec{x}) = \{\vec{w} \in \mathbb{R}^m : \vec{w}_i = \sup\{g(\vec{x}, \vec{v})_i^+ : \vec{v} \in \mathcal{V}\}\}. \quad (3)$$

The PB is a multidimensional non-negative measure of the worst constraint violation value possible of a optimization problem belonging to the OPUU family. It is a measure that gives the uncertainty protection in matters of the feasibility. It considers the intensity of the feasibility violation of a given OPUU solution, which is not done by the constraint set violation probability measure. In that sense, the PB is an innovative measure. However it does not induce an order on the solutions set.

Let $\eta : \mathbb{R}^n \rightarrow [0, 1]$ be the measure called *hardness* defined as:

$$\eta(\vec{x}) = \sum_{j=1}^m p_j \cdot \int_0^1 Pr\left(\alpha \cdot \beta^+(\vec{x})_j \geq g(\vec{x}, \vec{v})_j^+ : \vec{v} \in \mathcal{V}\right) d\alpha. \quad (4)$$

where $\sum_{j=1}^m p_j = 1, p_j \geq 0$ represents the decision maker's importance scale.

Hardness is a non-negative quality measure with 1 as the upper bound. It is a weighted comparison of how distant the worst case scenario constraint value is from all possible constraint values. It indicates how difficult is to violate the constraints of an OPUU in terms of the uncertain set and the impact of uncertain parameters on each constraint.

The mathematical properties of the hardness function led to some interesting results with regard to robustness.

Proposition 1. Consider an OPUU defined as presented in (1). If a solution $\vec{x} \in \mathbb{R}^n$ is such that $g(\vec{x}, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}$, then $\eta(\vec{x}) = 1$.

Proposition 1. If a solution $\vec{x} \in \mathbb{R}^n$ is such that $g(\vec{x}, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}$, its brittleness is zero:

$$\begin{aligned} \beta(\vec{x}) &= \{\vec{w} \in \mathbb{R}^m : \vec{w}_j = \sup\{g(\vec{x}, \vec{v})_j^+, \vec{v} \in \mathcal{V}\}\} \\ &= \{\vec{w} \in \mathbb{R}^m : \vec{w}_j = 0\} \Rightarrow \beta(\vec{x}) = 0 \end{aligned}$$

Note that $\beta(\vec{x}) = 0 \Rightarrow \Pi(\vec{x}, \alpha) = \vec{1}, \forall \alpha \in [0, 1]$. Therefore:

$$\eta(\vec{x}) = \sum_{j=1}^m p_j \cdot \int_0^1 \Pi_j(\vec{x}, \alpha) d\alpha = \sum_{j=1}^m p_j = 1$$

□

Proposition 3. Let $\vec{v}^* \in \mathcal{V}$ be the uncertain parameter for which $\vec{x}^*$ is the optimal solution for the optimization problem belonging to OPUU (1):

$$\min f(\vec{x}) \quad (5)$$

$$\text{S.t. } g(\vec{x}, \vec{v}^*) \leq 0 \quad (6)$$

Suppose that does not exist a robust solution for the OPUU (1). That implies:

1. There is no $\vec{x} \in \mathbb{R}^n$ such that $\forall \vec{v} \in \mathcal{V}, g(\vec{x}, \vec{v}) \leq 0$;
 - That would be a contradiction with the fact that $g(\vec{x}^*, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}$.
2. There is no $L \in \mathbb{R}$ such that $\forall \vec{x} \in \mathbb{R}^n$, where $g(\vec{x}, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}, f(\vec{x}) > L$;
 - Since $\vec{x}^*$ is a feasible solution for OPUU (1), there is a feasible solution $\vec{w} \neq \vec{x}^*$, such that $f(\vec{w}) < f(\vec{x}^*)$. If there was no such $\vec{w}$, then it is enough to define $L = f(\vec{x}^*) - 1$, which would be a contradiction. Hence, since $\vec{w}$ is a feasible solution for OPUU (1), it is necessary that $g(\vec{w}, \vec{v}^*) \leq 0$, which implies the absurd that $f(\vec{x}^*) \leq f(\vec{w})$.
3. There is $L = \inf_{\vec{x}} \{f(\vec{x}) | g(\vec{x}, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}\}$ but does not exists $\vec{x}$ feasible for OPUU (1) such that $f(\vec{x}) = L$.
 - By definition, $\forall \varepsilon > 0$, exists $\phi \in \{f(\vec{x}) | g(\vec{x}, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}\}$ such that $L < \phi \leq L + \varepsilon$. Let $\varepsilon = \frac{L - f(\vec{x}^*)}{2}$ and note that $L + \varepsilon < f(\vec{x}^*)$. Therefore, exists $\vec{w} \in \mathbb{R}^n$ such that $g(\vec{w}, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}$ and $L < f(\vec{w}) \leq L + \varepsilon < f(\vec{x}^*)$. However, analogously to the previous point, $g(\vec{w}, \vec{v}^*) \leq 0$, which implies that $f(\vec{x}^*) \leq f(\vec{w})$.

Let $\vec{w} \in \mathbb{R}^n$ be a robust solution for OPUU (1). Therefore, $f(\vec{x}^*) \geq f(\vec{w})$ and by the same argument above, $f(\vec{x}^*) \leq f(\vec{w})$, which implies that $\vec{x}^*$ is a robust solution for OPUU (1). $\square$

Proposition 4. Since $\eta(\vec{x}^*) = 1$, then:

$$\sum_{j=1}^m p_j \cdot \int_0^1 Pr \left(\alpha \cdot \beta_j(\vec{x}^*) \geq g(\vec{x}^*, \vec{v}_l)_j^+ : \vec{v}_l \in \mathcal{V} \wedge l \in \{1, \dots, L\} \right) d\alpha = 1. \quad (7)$$

Let $J \subset \mathbb{N}$ be the set of indexes $j \in J$ where $\beta_j(\vec{x}^*) \neq 0$. J must be the empty set, since:

If $J = \emptyset$: Notice that $g_j(\vec{x}^*, \vec{v}) \leq \beta_j(\vec{x}^*) = 0, \forall \vec{v} \in \mathcal{V}$. Therefore, by Proposition 3, $\vec{x}^*$ is a robust solution for the OPUU (1).

If $J \neq \emptyset$, for every $j \in J$:

$$\alpha \cdot \beta_j(\vec{x}^*) \geq g(\vec{x}^*, \vec{v}_l)_j^+ \iff \alpha \in \left[\frac{g(\vec{x}^*, \vec{v}_l)_j^+}{\beta_j(\vec{x}^*)}, 1 \right] \quad (8)$$

For sake of simplicity, let $w_{jl} = \frac{g(\vec{x}^*, \vec{v}_l)_j^+}{\beta_j(\vec{x}^*)}$. This turns equation (7) into:

$$\eta(\vec{x}^*) = \sum_{j \in J} p_j \cdot \int_0^1 Pr(\alpha \in [w_{jl}, 1] : l \in \{1, \dots, L\}) d\alpha = 1 \quad (9)$$

By the additive property of probability,

$$\eta(\vec{x}^*) = \sum_{j \in J} p_j \cdot \sum_{l=1}^L \int_{w_{jl}}^1 Pr(\alpha \in [w_{jl}, 1]) d\alpha = 1. \quad (10)$$

Suppose that each $\vec{v}_l \in \mathcal{V}, l \in \{1, \dots, L\}$ has a unknown probability $\pi_l \in (0, 1]$, with $\sum_{l=1}^L \pi_l = 1$, of become reality. Then,

$$\begin{aligned} \eta(\vec{x}^*) &= \sum_{j \in J} p_j \cdot \sum_{l=1}^L \left(\int_{w_{jl}}^1 Pr(\alpha \in [w_{jl}, 1] : j \in J) d\alpha \right) \\ &= \sum_{j \in J} p_j \cdot \sum_{l=1}^L \left(\int_{w_{jl}}^1 \pi_l d\alpha \right) = \sum_{j \in J} p_j \cdot \sum_{l=1}^L \pi_l (1 - w_{jl}) = 1, \end{aligned} \quad (11)$$

which results in,

$$\begin{aligned} \eta(\vec{x}^*) &= \sum_{j \in J} p_j \cdot \left(\sum_{l=1}^L \pi_l - \sum_{l=1}^L \pi_l w_{jl} \right) = 1 \\ &= \sum_{j \in J} p_j \cdot \left(1 - \sum_{l=1}^L \pi_l w_{jl} \right) = 1 \end{aligned} \quad (12)$$

But $\sum_{l=1}^L \pi_l w_{jl} = \sum_{l=1}^L \pi_l \frac{g(\vec{x}^*, \vec{v}_l)_j^+}{\beta_j(\vec{x}^*)} > 0, \forall j \in J$, which is a contradiction with the fact that $J \neq \emptyset$. And finally:

$$\beta_j(\vec{x}^*) = 0, \forall j \in \{1, \dots, m\} \Rightarrow g(\vec{x}^*, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V} \quad (13)$$

Therefore, by Proposition 3, $\vec{x}^*$ must be the robust solution of OPUU (1). $\square$

Proposition 2. If $\vec{x}^*$ is the robust solution for (1), then $\eta(\vec{x}^*) = 1$.

That is, the hardness measure is maximum for the robust solution of an OPUU. But what it takes to define a robust solution given that its robustness is maximum?

Proposition 3. Consider the vector $\vec{x}^* \in \mathbb{R}^n$, a function $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$ and $\mathcal{V} \subset \mathbb{R}^q$. Suppose that $g(\vec{x}^*, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}$. If $\vec{x}^*$ is an optimal solution for a optimization problem belonging to OPUU (1) then $\vec{x}^*$ is a robust solution of OPUU (1).

Note that Proposition 3 presents a necessary but not a sufficient condition to characterize a robust solution. Consider the following OPUU:

$$\max_{x,y} y \quad (14)$$

$$\text{S.t. } y + (x - 1) \cdot \left(\frac{\sin \theta}{1 - \cos \theta} \right) \leq 0 \quad \forall \theta \in \left[\frac{\pi}{4}; \frac{3\pi}{4} \right] \quad (15)$$

$$y - (x + 1) \cdot \left(\frac{\sin \theta}{\cos \theta + 1} \right) \leq 0 \quad \forall \theta \in \left[\frac{\pi}{4}; \frac{3\pi}{4} \right] \quad (16)$$

Its robust solution is $(x^*, y^*) = (0, -1 + \sqrt{2})$ but every problem belonging on the family has a optimal solution in the form $(x^*, y^*) = (\cos \theta, \sin \theta)$.

Proposition 4. *If the uncertain set $|\mathcal{V}| = L > 1$ and:*

1. $\vec{x}^* \in \mathbb{R}^n$ is such that $g(\vec{x}^*, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}$;
2. Does not exist $\vec{w} \in \mathbb{R}^n$ with $f(\vec{w}) < f(\vec{x}^*)$ and $g(\vec{w}, \vec{v}) \leq 0, \forall \vec{v} \in \mathcal{V}$;
3. $\eta(\vec{x}^*) = 1$

Then $\vec{x}^*$ is a robust solution of OPUU (1).

Now, we will illustrate both measures using a modified version of the example from Birge and Louveaux [Birge e Louveaux, 2011] page 4. The advantage of the hardness measure for this example over the other measure will be presented.

3. The farmer problem

A farmer cultivates wheat, corn and sugar beets on an area that has 500 acres of land. The farmer also has cattle to take care of and needs at least 240 tons (T) of corn and 200 tons of wheat to feed them. There is a wholesaler where he or she can buy wheat and corn if he or she can't produce those quantities for 140% of the farmer's selling price. Over the last ten years, the mean selling prices for wheat and corn were \$170/T and \$150/T respectively. The government knows that the sugar beet crop is usually the most profitable one. In order to control the prices and production the government fixes at 6,000 tons the amount of sugar beet that can be sold at \$36/T and the excess of the quota can be sold at \$10/T. The farmer knows that his/her land can produce roughly 2.5T, 3T and 20T per acre for wheat, corn, and sugar beets, respectively and wants to know the amount of land that should be planted in wheat, corn and sugar beets crop, the total weight of production of this products, the amount of cattle feed that he or she will need to buy from the wholesaler. The farmer's main objective is to maximize profits.

The fact that the farmer knows "roughly" that the production efficiency of his/her land and the fact that he or she will plan all his/her work based on this, indicates that, from the data, there is a level of risk. The data indicates that there are quite different yields for the same crop over different years mainly because of changing weather conditions. Yields varying 20 to 25% above or below the mean yield are not unusual, showing that it is useful to address the uncertainty associated with this information.

The farmer problem under uncertainty can be formulated as presented in [Birge e Louveaux, 2011]. Let each parameter assume values in a symmetric interval around the mean with magnitude of 40% of the nominal value. The uncertain set $\mathcal{V} \subset \mathbb{R}^3$ can be defined as $\mathcal{V} = [2, 3] \times [2.4, 3.6] \times [16, 24]$.

Suppose the farmer wants to find a solution that will stay feasible for certain combinations of the worst case parameters. It is plausible to assume that not everything can go wrong at the same time. Bertsimas and Sim [Bertsimas e Sim, 2004] proposed a robust optimization model which takes into account an uncertainty budget. Let J_i be the set of uncertain parameters on the i -th constraint. A robust parameter $\Gamma_i \in [0, |J_i|]$ is defined for each constraint i that is affected by uncertainty. That is, for the farmer's problem, $\Gamma_i \in [0, 1], \forall i \in \{1, 2, 3\}$. For each possible $\Gamma \in [0, 1]^3 \subset \mathbb{R}^3$ value, the mathematical model presented below, represents the deterministic optimization problem where the solution will remain feasible even if $\lfloor \Gamma_i \rfloor$ uncertain parameters assume their maximum values. The amount of uncertain parameters that are allowed to assume its maximum value is the greatest integer that is less than Γ_i .

$$\min \quad 150 \cdot x_1 + 230 \cdot x_2 + 260 \cdot x_3 + 238 \cdot y_1 - 170w_1 \quad (17)$$

$$210 \cdot y_2 - 150 \cdot w_2 - 36 \cdot w_3 - 10w_4 \quad (18)$$

$$\text{S. t.} \quad x_1 + x_2 + x_3 \leq 500 \quad (19)$$

$$2.5 \cdot x_1 + y_1 - w_1 - z_1 \cdot \Gamma_1 - p_{21} \geq 200 \quad (20)$$

$$3 \cdot x_2 + y_2 - w_2 - z_2 \cdot \Gamma_2 - p_{32} \geq 240 \quad (21)$$

$$w_3 + w_4 + z_3 \cdot \Gamma_3 + p_{43} \leq 20 \cdot x_3 \quad (22)$$

$$w_3 \leq 6000 \quad (23)$$

$$z_1 + p_{21} \geq 0.5 \cdot r_1 \quad (24)$$

$$z_2 + p_{32} \geq 0.6 \cdot r_2 \quad (25)$$

$$z_3 + p_{43} \geq 4 \cdot r_3 \quad (26)$$

$$-r_j \leq x_j \leq r_j, \forall j \in \{1, 2, 3\} \quad (27)$$

$$x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0 \quad (28)$$

$$z_1, z_2, z_3, r_1, r_2, r_3, p_{21}, p_{32}, p_{43} \geq 0 \quad (29)$$

On the farmer problem, if $\Gamma_1 = 1$ then the optimal solution for (18) - (29) will be protected from scenarios where the wheat production efficiency is the worst possible.

4. Computational Methodology

The computational tests are based on evaluating the two measures presented here in a simulation environment. The process is done in two steps. The first is to construct a reduced uncertain set $RS(\mathcal{V}) \subset \mathcal{V}$, since the simulation is composed of a finite number of scenarios. These were performed using Monte Carlo simulation of $N = 100,000$ points with uniform distribution on the original uncertain set $\mathcal{V} = [2, 3] \times [2.4, 3.6] \times [16, 24]$.

Then, the second step is to calculate the two measures based on the simulated points that were used to construct the reduced uncertain set. For the constraint set violation probability measure ($\zeta(\vec{x})$), it is known that, by Borel's Proposition (*Law of large numbers*) [Gnedenko, 2017], $\hat{\zeta}(\vec{x}, N)$ defined by equation (30) below is a good approximation when N is large enough.

$$\hat{\zeta}(\vec{x}, N) = \frac{\sum_{i=1}^N \delta \left(\sum_{j=1}^m g(\vec{x}, \vec{v}_i)_j^+ \right)}{N}, \quad (30)$$

where $\delta(y) = 1$ if $y > 0$ and 0 otherwise.

The hardness approximation is divided in two parts. The first is the definition of the approximation for the PB measure (β^+), which is straightforward from PB's definition when the uncertain set ($\mathcal{V}$) is replaced with the reduced one ($RS(\mathcal{V})$). For the second part of the approximation, it is necessary to discretize the range of the values of α in order to numerically calculate the integral. Here $\varrho = 100$ points evenly distributed along the interval $[0, 1]$ were used. The final expression that represents the *approximate hardness* ($\hat{\eta}$) is presented below in equation (31),

$$\hat{\eta}(\vec{x}, N) = \sum_{j=1}^m p_j \cdot \left(\sum_{i=1}^{\varrho} \sum_{i=1}^N \delta \left(\frac{i}{\varrho} \cdot \hat{\beta}^+(\vec{x})_j - g(\vec{x}, \vec{v}_i)_j^+ \right) \cdot \frac{1}{\varrho \cdot N} \right). \quad (31)$$

Table 1: Five different Γ values were used to find solutions for the Farmer problem. Hardness can be used to identify when the trade of objective function value for uncertainty protection really pays of.

	x_1	x_2	x_3	y_1	y_2	w_1	w_2	w_3	w_4	$f(x)$	$\zeta(x)$	$\eta_E(x)$	$\eta_W(x)$	$\eta_C(x)$
R.O. 1	120	80	300	0	0	100	0	6000	0	-118600	87.5%	85.1%	75.3%	75.3%
R.O. 2	100	100	300	0	0	0	60	6000	0	-109000	75.3%	90.1%	100%	75.2%
R.O. 3	200	0	300	0	240	300	0	6000	0	-108600	74.8%	90.1%	75.2%	100%
R.O. 4	80	80	340	0	0	0	0	5440	0	-77040	75.1%	90.1%	75.2%	75.2%
R.O. 5	100	25	375	0	180	0	0	6000	0	-59950	0.00%	100%	100%	100%

All the values reported in the tables will be labeled without the hat above the character that represents the respective measure for the sake of simplicity. All values were obtained using the approximations as described.

5. Evaluating the feasibility measures

Five different Γ values were used to find solutions for the Farmer problem. R.O. 1 has $\Gamma = (0, 0, 0)$ and is equivalent to consider the scenario where all uncertainty parameters assume the average value. R.O. 2 is associated with $\Gamma = (1, 0, 0)$, R.O. 3 has $\Gamma = (0, 1, 0)$ and R.O. 4 has $\Gamma = (0, 0, 1)$. They are scenarios where wheat, corn and sugar beets crops are the worst possible in terms of efficiency, respectively and all other crops have average efficiency. Finally, R.O. 5 has $\Gamma = (1, 1, 1)$ and is the worst case scenario when every crop has its worst production per unit of land area.

Three different farmer's preferences are considered to evaluate the hardness of each solution found. The first preference η_E shows a equal preference of being protected from crop efficiency uncertainty of wheat, corn and sugar beets. The second preference η_W reflects an worry by the farmer of being protected mostly from wheat production uncertainty. The third preference η_C is about corn production deficiency.

Table 1 show the solution found for scenario R.O 1 through scenario R.O. 5. Hardness can be used to identify when the trade of objective function value for uncertainty protection really pays of. Constraint set violation probability indicates that 1,5% more uncertainty protection worth 0,03% of the objective function. Hardness indicates that R.O. 2 is better then R.O. 3 and R.O. 4, unless the farmer has total preference of not been exposed to corn uncertainty production. That is, constraint set violation probability measure indicate to the farmer that is better not to produce corn and buy from a wholesaler in order to feed the cattle. Hardness, instead, indicates that produce wheat and corn to feed the cattle and sell the remaining food will be a more effective policy.

Here we can see that the hardness measure can reveal interesting information about the uncertainty protection of a given solution. The hardness measure analyzed against the objective function value can show the trade-off between the uncertainty protection and the solution quality, which is valuable information when one decides to consider uncertainty during the optimization phase. That is, the hardness measure is indeed a valuable tool for evaluating and classifying solutions for problems that are faced with parametric uncertainty.

6. Remarks and Final Conclusions

This research defined a new measure for optimization under uncertainty. This new measure, called *hardness*, is a relative comparison between the worst case scenario constraint value and all constraint values possible for uncertain parameters belonging to a uncertainty set. One can use it to identify the comparative protection that is achieved by the method of interest. The hardness is a positive measure with maximum value of 1, representing that the solution is robust and can

be weighted by the decision maker preferences. The measure usefulness was verified during the process of analyzing solutions for a classic problem from the literature. It brought more depth and flexibility to the decision maker, fulfilling a need in the literature for quality measures for optimization problems under uncertainty.

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