LII Simpósio Brasileiro de Pesquisa Operacional
João Pessoa - PB, 3 a 5 de novembro de 2020

INDEPENDENCE GAP IN GRAPHS OF GIRTH AT LEAST 6

Márcia R. Cappelle
Instituto de Informática – Universidade Federal de Goiás
Alameda Palmeiras, Quadra D, Campus Samambaia, 74001-970, Goiânia, GO, Brasil
marcia@inf.ufg.br

Marcelo C. Dias
Instituto de Informática – Universidade Federal de Goiás
Alameda Palmeiras, Quadra D, Campus Samambaia, 74001-970, Goiânia, GO, Brasil
marcelocdias93@gmail.com

Vinícius Gabriel Santos
Instituto de Informática – Universidade Federal de Goiás
Alameda Palmeiras, Quadra D, Campus Samambaia, 74001-970, Goiânia, GO, Brasil
vinicius.gabriel.px@gmail.com

RESUMO

A diferença de independência de um grafo G é a diferença entre os tamanhos de um maior e de um menor de seus conjuntos independentes maximais. Este conceito é uma relaxação do conceito de grafos bem-cobertos, que são aqueles que possuem todos os seus conjuntos independentes maximais de mesmo tamanho. Grafos bem-cobertos têm diferença de independência zero. Apresentamos caracterizações de alguns grafos com diferença de independência pelo menos 1 que têm cintura pelo menos 6 incluindo grafos com diferença de independência $r-1$, para $r \geq 2$, com r cardinalidades diferentes, mas consecutivas, de conjuntos independentes maximais. Apresentamos resultados sobre a quantidade de árvores com valores específicos máximo e mínimo de conjuntos independentes maximais.

PALAVRAS CHAVE. Conjuntos independentes, grafos bem-cobertos, cintura do grafo.

TAG – Teoria e Algoritmos em Grafos

ABSTRACT

The independence gap of a graph G is the difference between the maximum and the minimum sizes of a maximal independent set in G. This concept is a relaxation of the concept of well-covered graphs, which are graphs with all maximal independent sets of the same size. Well-covered graphs have independence gap zero. We present characterizations of some graphs with independence gap at least 1 that are of girth at least 6, including graphs with independent gap $r-1$, for $r \geq 2$, with r distinct and consecutive sizes of maximal independent sets. We present results related to the number of trees with specific maximum and minimum sizes of maximal independent sets.

KEYWORDS. Independent sets. Well-covered graphs. Girth.

GTA – Graph Theory and Algorithms

Galoá { Este trabalho foi publicado utilizando o **Galoá** proceedings

1. Introduction

We consider finite, simple, and undirected graphs. For a graph G, the vertex set and the edge set are denoted $V(G)$ and $E(G)$, respectively. We use standard notation and terminology. See Bondy and Murty [1976] for graph-theoretic terms not defined here.

A subset $D \subseteq V(G)$ *dominates* a subset $S \subseteq V(G)$ if every vertex in S has a neighbor in D. If D dominates $V(G)$, then D is a *dominating set* of G. A subset $I \subseteq V(G)$ is *independent* if no two vertices in I are adjacent. An independent set I of G is *maximal* if it is a dominating set and it is *maximum* if G has no independent set J with $|J| > |I|$. For a graph G we denote by $\alpha(G)$ the maximum size of an independent set in G and by $i(G)$ the minimum size of a maximal independent set in G.

A graph is *well-covered* if all its maximal independent sets have the same size. These graphs were introduced by Plummer [1970] and have been widely studied in the literature. See, for example, the survey papers by Plummer [1993] and Hartnell [1999]. Recognizing well-covered graphs is a co-$\mathcal{NP}$-Complete problem, as proven independently by Chvátal and Slater [1993] and Sankaranarayana and Stewart [1997].

On the other hand, if a graph is not well-covered, it is natural to know how far it is from being well-covered. Several new concepts have arisen generalizing the concept of well-covered graphs. Finbow et al. [1994] define the set $\mathcal{M}_r$, for every positive integer r, to be the set of graphs that have maximal independent sets of exactly r different sizes. With this notation, $\mathcal{M}_1$ is exactly the set of all well-covered graphs. It was proved by Caro et al. [1996] that recognizing graphs in $\mathcal{M}_r$ is a co-$\mathcal{NP}$-Complete problem. If a graph is a member of $\mathcal{M}_r$, where the distinct values of its maximal independent sets are consecutive, then it is also a member of $\mathcal{I}_r$, defined by Barbosa and Hartnell [1998].

Many results concerning independence consider the *girth* of a graph, which is the length of its smallest cycle. If a graph has no cycle, its girth is by definition equal to infinity. A *tree* is a connected graph with no cycles. Well-covered graphs of girth at least 5 were characterized by Finbow et al. [1993]. Graphs of girth at least 8 belonging to $\mathcal{M}_2$ were characterized by Finbow et al. [1994].

Another concept which generalizes the well-covered graphs in a different way is called *independence gap*, which is the difference between the maximum and minimum sizes of a maximal independent set in G, that is, the difference $\alpha(G) - i(G)$. For a graph G, we denote its independence gap by $\mu_\alpha(G)$. Therefore, well-covered graphs have independence gap equal to zero. Graphs that have independence gap equal to one are also called *almost well-covered graphs* and are exactly the set of graphs in $\mathcal{I}_2$. An efficient characterization of a subclass of almost well-covered graphs of girth at least 6 was presented by Ekim et al. [2018], as well as a polynomial-time algorithm for recognizing almost well-covered graphs that do not have cycles with sizes 3, 4, 5, and 7. Ekim et al. [2020] presented results concerning the independence gap of graphs focusing on perfect graphs. Cappelle and Rautenbach [2013] showed a tight upper bound on the independence number of graphs of independence gap equal to k, where $k \geq 0$.

When the restrictions on the graphs concern girth and minimum degree, for some values of r, there are only cycles belonging to $\mathcal{M}_r$. Finbow and Hartnell [1983] proved that C_7 is the only well-covered connected graph of minimum degree at least 2 and girth at least 6. Similarly, Finbow et al. [1994] proved that the cycles C_8, C_9, C_{10}, C_{11}, and C_{13} are the only graphs in $\mathcal{M}_2$ of minimum degree at least 2 and girth at least 8. For $r \geq 4$, Hartnell and Rall [2013] proved that every graph in $\mathcal{M}_r$ of minimum degree at least 2 and girth at least $6r - 6$ is a cycle. Barbosa et al. [2013] proved that, for integers r and d with $r \geq 2$ and $d \geq 3$, $\mathcal{M}_r$ contains only finitely many

Galoá { Este trabalho foi publicado utilizando o **Galoá** proceedings

connected graphs of minimum degree at least 2, maximum degree at most d, and girth at least 7. When graphs may have leaves, $\mathcal{M}_r$ contains infinitely many such graphs.

A vertex is said to be of *type* r if it is adjacent to exactly r leaves. We consider $r \geq 2$. We show that graphs of girth at least 4 with $\mu_\alpha(G) = r - 1$ have at most 2 vertices of type r. We characterize some graphs of girth at least 6 with $\mu_\alpha(G) = r - 1$ and exactly two vertices of type r, generalizing results by Ekim et al. [2018] for almost well-covered graphs. As a consequence of this result, we show that for girth at least 6, only graphs having girth exactly 6 in $\mathcal{I}_3$ have exactly two vertices of type r. Besides that, we present a procedure to build graphs in $\mathcal{I}_r$ from other graphs which also belong to $\mathcal{I}_r$. We characterize trees in $\mathcal{I}_3$ that contain exactly one vertex of type 3 and exactly two vertices of type 2. We prove some results related to the number of trees with specific maximum and minimum sizes of maximal independent sets.

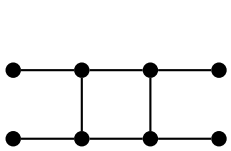
The remainder of this paper is organized as follows. In Section 2, we present further definitions and some basic results. In Section 3, we present the results concerning the independence gap in graphs of girth at least 6, along with some consequences for the set $\mathcal{I}_r$. In Section 4, we show the results related to trees.

2. Definitions and preliminary results

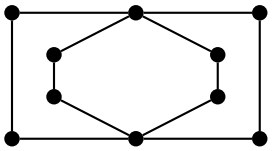
For a vertex u of G, the neighborhood, the closed neighborhood, and the degree are denoted $N_G(u)$, $N_G[u]$, and $d_G(u)$, respectively. For a set U of vertices of G, $N_G[U] = \bigcup_{u \in U} N_G[u]$.

As usual, we denote by P_n, C_n, and K_n the path, the cycle and the complete graph of order n, respectively. A vertex of degree 1 is a *leaf* and a *support vertex* is a vertex adjacent to a leaf. A vertex is called *internal* if it is not a leaf. A *pendant edge* is an edge between a support vertex and a leaf. We assign a non-negative integer, called the type of u, to some vertices u of G. Internal vertices adjacent to exactly k leaves will be of type k. Note that a leaf in a component other than K_2 is not assigned to any type.

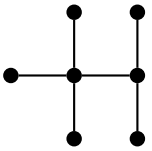
A *perfect matching* in a graph is a set of pairwise disjoint edges such that every vertex of the graph is an endpoint of one of them. We abuse notation and write $X - a$, when X is a set and a is a possible member of X, instead of $X - \{a\}$. If $X \subseteq V(G)$, $G - X$ denotes the subgraph of G obtained by removing vertices in X and edges incident to them in G. We denote the set of positive integers between p and q, inclusive, for $[p, q]$.



(a) $\text{miss}(G_1) = \{4\}$.



(b) $\text{miss}(G_2) = \{2, 4, 5\}$.



(c) $\text{miss}(G_3) = \{3, 4, 5\}$.

Figure 1: Graph G_1 is well-covered, and $\mu_\alpha(G_1) = 0$; $G_2 \in \mathcal{M}_3$, but $G_2 \notin \mathcal{I}_3$, and $\mu_\alpha(G_2) = 3$; $G_3 \in \mathcal{I}_3$, therefore $G_3 \in \mathcal{M}_3$, and $\mu_\alpha(G_3) = 2$.

For short, we write MIS instead of maximal independent set. For a graph G, $\text{miss}(G) = \{|I| \ : \ I \text{ is a MIS of } G\}$, that is, it is the set with all sizes of MIS of G. In Figure 1, we present some examples of graphs and the sets to which they belong concerning their different sizes of MIS. Note that if a graph G is a member of $\mathcal{I}_r$, then it belongs to $\mathcal{M}_r$ and $\mu_\alpha(G) = r - 1$. However, if $\mu_\alpha(G) = r - 1$, not necessarily it belongs to $\mathcal{M}_r$. Similarly, if G belongs to $\mathcal{M}_r$, not necessarily it belongs to $\mathcal{I}_r$.

Finbow et al. [1994] showed that if G is a graph in $\mathcal{M}_r$, then for every independent set I of G, $G - N_G[I]$ belongs to $\mathcal{M}_s$ for some $s \leq r$. The following lemma is a straightforward implication of this result that will be used in some proofs.

Lemma 1. *(Ekim et al. [2018]) For an independent set I of a graph G, $\mu_\alpha(G - N_G[I]) \leq \mu_\alpha(G)$.*

We consider connected graphs, since the following lemma reduces the problem of determining the independence gap of a graph to its components.

Lemma 2. *(Ekim et al. [2018]) If G is a graph with components $H_1, \ldots, H_k$, then $\mu_\alpha(G) = \sum_{j=1}^{k} \mu_\alpha(H_j)$.*

The following result, which is a characterization of well-covered graphs of girth at least 6, is used in some proofs.

Theorem 1. *(Finbow et al. [1993]) Let G be a connected graph of girth at least 6 isomorphic to neither C_7 nor K_1. Then G is well-covered if and only if its pendant edges form a perfect matching.*

3. Results on independence gap

Ekim et al. [2018] investigated graphs in $\mathcal{I}_2$, which are also called almost well-covered graphs. They considered graphs of girth at least 6. If G is almost well-covered having girth at least 6, then G has at most two vertices of type 2. The authors did not present a complete characterization for these graphs, but they characterized the ones that have exactly two vertices of type 2 and the ones with only one vertex of type 2. We use some arguments from Ekim et al. [2018] to generalize results related to the independence gap in a given graph G, considering $\mu_\alpha(G) \leq r - 1$, for $r \geq 2$. We show the possible sizes of MIS for graphs of girth at least 6 with $\mu_\alpha(G) \leq r - 1$ having exactly two vertices of type r. We use this result to characterize graphs in $\mathcal{I}_3$ of girth at least 6 having exactly two vertices of type 3 and to show that graphs in $\mathcal{I}_r$ of girth at least 7 have at most one vertex of type r.

The following result bounds the number of leaves a vertex of a graph G can have considering that $\mu_\alpha(G) \leq r - 1$.

Lemma 3. *(Ekim et al. [2018]) If G is a graph with $\mu_\alpha(G) \leq r - 1$, then every internal vertex of G is adjacent to at most r leaves.*

In this section, we denote G_i the subgraph of G induced by internal vertices of G that are type i. For instance, G_0 is the subgraph of G that containing internal vertices with no leaves.

Lemma 4. *If G is a graph with $\mu_\alpha(G) = r - 1$, for some $r \geq 2$, then G_r induces a complete graph.*

Proof. Let G be a graph such that $\mu_\alpha(G) = r - 1$, for some $r \geq 2$. Toward a contradiction, suppose that G has two vertices of type r, say u_1 and u_2, with sets of leaves F_1 and F_2, respectively, and u_1 and u_2 are not adjacent. Hence there is an MIS I of G that contains both, u_1 and u_2. Let $I' = (I - \{u_1, u_2\}) \cup F_1 \cup F_2$. Extend I' to a maximum independent set of G. It follows that $|I'| - |I| \geq 2r - 2$. Since $r \geq 2$, we conclude that $\mu_\alpha(G) > r - 1$. □

Lemma 4 immediately implies that if $\mu_\alpha(G) = r - 1$ for some $r \geq 2$ and G has girth at least 4, G has at most two vertices of type r. From now on, we consider graphs of girth at least 6.

Lemma 5. *Let G be a graph of girth at least 6 such that $\mu_\alpha(G) = r - 1$, for some $r \geq 1$. If G has a vertex u of type r, then vertex u does not have a neighbor in G_0.*

Proof. Let G be as described. Suppose that G has a vertex u of type r. For contradiction, suppose that u has a neighbor v in G_0. Let I be the set of vertices that are, at same time, at distance 3 from t and distance 2 from v. By the girth restriction, I is independent. Let $G' = G - N_G[I]$. Observe that in G' vertex v is a leaf of u that has $r+1$ leaves. By Lemma 3, $\mu_\alpha(G') > r-1$ and, by Lemma 1, $\mu_\alpha(G) > r-1$. □

Lemma 6. *Let G be a graph of girth at least 6 such that $\mu_\alpha(G) = r-1$, for some $r \geq 2$. If G has exactly two vertices of type r, then any other support vertex of G is of type 1.*

Proof. Let G be as described. Suppose that G has two support vertices, say u_1 and u_2, with sets of leaves F_1 and F_2, respectively, such that $|F_1| = |F_2| = r$. By Lemma 4, u_1 and u_2 are adjacent. For a contradiction, suppose that G has another support vertex, say x, which has the leaves in the set F_3 with $|F_3| \geq 2$. Because of the girth restriction, x is not adjacent to at least one of u_1 and u_2. Without loss of generality, suppose that x is not adjacent to u_2. Let I be an MIS of G that contains both x and u_2. Hence $F_1 \subseteq I$. Let $I' = I - \{x, u_2\} \cup F_2 \cup F_3$. Extend I' to a maximum independent set of G. Since $|F_3| \geq 2$, it follows that $|I'| - |I| \geq |F_3| + r - 2 \geq r$, which contradicts that $\mu_\alpha(G) = r-1$. □

The following result is a generalization of Theorem 3.3 by Ekim et al. [2018] for graphs G of girth at least 6 with $\mu_\alpha(G) = 1$. We adapt their proof considering $\mu_\alpha(G) \geq 1$. Additionally, we present the different sizes of MIS of G.

Theorem 2. *Let $r \geq 2$ and G be a connected graph of girth at least 6, with exactly two vertices u_1 and u_2 of type r, and with no type k vertices for $k \geq r+1$. Then $\mu_\alpha(G) = r-1$ if and only if u_1 and u_2 are adjacent, any other support vertex of G is type 1, and one of the following two conditions holds:*

(i) $V(G_0) = \emptyset$;

(ii) $G_0 \cong K_2$, neither of u_1 and u_2 has a neighbor in G_0, and the two vertices of G_0 are of degree 2 in G and are contained in an induced 6-cycle containing u_1 and u_2.

Moreover, if $V(G_0) = \emptyset$, then $\mathrm{miss}(G) = \{|V(G_1)| + r + 1, |V(G_1)| + 2r\}$ *otherwise* $\mathrm{miss}(G) = \{|V(G_1)| + r + 2, |V(G_1)| + 2r, |V(G_1)| + 2r + 1\}$.

Proof. Let G be as described and let F_1 and F_2 be the sets of leaves, respectively, of vertices u_1 and u_2. Since both are type r, $|F_1| = |F_2| = r$. First suppose $\mu_\alpha(G) = r-1$. By Lemmas 5 and 6, the other neighbors of vertices u_1 and u_2 are vertices of type 1. By Lemma 4, u_1 and u_2 are adjacent. Suppose $V(G_0) \neq \emptyset$; Let $L_1 = N_G(u_1) - (F_1 \cup \{u_2\})$ and $L_2 = N_G(u_2) - (F_2 \cup \{u_1\})$. By the girth restriction, $L_1 \cup L_2$ is an independent set. Let L_i' the set of leaves adjacent to vertices of L_i, $i = 1, 2$. Note that $|L_i| = |L_i'|$. Now, let $I = F_1 \cup F_2 \cup L_1' \cup L_2'$ and let $G' = G - N_G[I]$. The graph G' contains all vertices in G_0, and possibly other vertices in G_1, with their respective leaves. See Figure 2 for an illustration.

Claim 1. *Graph G' is well-covered.*

Proof of Claim 1. *Observe that $\alpha(G) = \alpha(G') + |I| = \alpha(G') + |L_1| + |L_2| + 2r$ and $i(G) = i(G') + |L_1| + |L_2| + r + 1$. If G' is not well-covered, then $i(G') < \alpha(G')$, which implies $\alpha(G) - i(G) \geq r$.*

Claim 2. *Graph G' has a perfect matching formed by its pendant edges.*

Proof of Claim 2. *Since G' is well-covered (by Claim 1) and the girth of G' is at least 6, by Theorem 1 every component of G' is isomorphic to K_1, C_7, or to a graph with a perfect matching formed by its pendant edges. Since the girth of G' is at least 6, it is easy to see that no component of G' is isomorphic to K_1. Suppose that G' has a component isomorphic to C_7. Let C be the 7-cycle formed by vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$, in this order. Since G is connected, we may assume without loss of generality that there exists a vertex $a \in L_1$ such that a is adjacent to v_1. By the girth restriction, a is not adjacent to any other vertex in C. Extend $\{v_3, v_6\}$ to an* MIS *J' in $G' - N_{G'}(a)$. Observe that $|J'| < \alpha(G')$ since J' contains exactly two vertices of C, while any other* MIS *of G' contains three vertices of C. Extend $J' \cup \{a, u_2\}$ to a maximum independent set J of G. It is easy to see that $|J| < \alpha(G') + |L_1| + |L_2| + r + 1 = \alpha(G) - r + 1$, contradicting that $\mu_\alpha(G) = r - 1$ and proving that no component of G' is isomorphic to C_7. Therefore, G' has a perfect matching formed by its pendant edges.*

Claim 3. *Every component of G_0 is isomorphic to K_2.*

Proof of Claim 3. *By Claim 2 G' has a perfect matching formed by its pendant edges. By definition of G', $V(G_0) \subseteq V(G')$. First, we show that G_0 does not have a component different from K_2, that is, every vertex of G_0, is a leaf in G'. Suppose to the contrary that there exists an edge of the perfect matching with endpoints z and z' such that $z, z' \in V(G_0)$ and that z' is a leaf in G'. Hence there exists a vertex $w \neq z'$ in G' such that z is adjacent to w. Note that either $w \in V(G_0)$ or $w \in V(G') \cap V(G_1)$. Without loss of generality, we may assume that there exists a vertex $a \in L_1$ such that z' is adjacent to a. The girth restriction implies that w and a are not adjacent vertices in G. Let I be an* MIS *in G containing $\{a, w, u_2\}$. It is easy to see that $|I \cap V(G')| < \alpha(G')$ since $z, z' \notin I$. Therefore, $|I| = |I \cap V(G')| + |L_1| + |L_2'| + r + 1 < \alpha(G) - r + 1$, contradicting that $\mu_\alpha(G) = r - 1$. This concludes the proof of Claim 3.*

Let z and z' be two adjacent vertices in G_0. Since, by Claim 3, the component of G_0 containing z and z' is a component of G_0 isomorphic to K_2, z and z' are vertices of G_0, each one of them has a neighbor in $L_1 \cup L_2$. We may assume, without loss of generality, that there exists a vertex $a \in L_1$ such that z' is adjacent to a. Furthermore, there is a neighbor b of z in $L_1 \cup L_2$. By the girth restriction, $b \in L_2$. Since the girth of G is at least 6, z and z' have no other neighbors in $L_1 \cup L_2$. Hence $d_G(z) = d_G(z') = 2$ and they are under a 6-cycle containing u_1 and u_2.

Next, we show that G_0 has exactly one component, that is, $V(G_0) = \{z, z'\}$. Suppose that it is not true and consider that $G_0 - \{z, z'\}$ has a pair of adjacent vertices, say w and w'. Since the component of G_0 containing w and w' is a component of G' isomorphic to K_2, and w, w' are internal vertices of G, each one of them has a neighbor in $L_1 \cup L_2$. These imply that $L_1 \cup L_2$ is an independent set of G that dominates $\{z, z', w, w'\}$. If I is an MIS of G containing $L_1 \cup L_2$, then $|I \cap V(G')| \leq \alpha(G') - r$, since $I \cap V(G')$ contains neither of vertices in $\{z, z', w, w'\}$, while any maximum independent set in G' must contain one of z and z', and one of w and w'. Recall that $\alpha(G) = \alpha(G') + |L_1| + |L_2| + 2r$. Therefore, $|I| < \alpha(G) - r + 1$, contradicting that $\mu_\alpha(G) = r - 1$.

Conversely, suppose $V(G_0) = \emptyset$. Hence $V(G)$ contains exactly the vertices u_1 and u_2, their leaves ($F_1 \cup F_2$), the vertices in G_1, and their leaves. Note that any MIS of G contains exactly one vertex of every pendant edge with one of its endpoints in G_1. Let I be an MIS of G. If $I \cap \{u_1, u_2\} = \emptyset$, I contains the leaves in $F_1 \cup F_2$ and $|I| = 2r + |L_1| + |L_2| + |V(G')|/2 = |V(G_1)| + 2r$. If $I \cap \{u_1, u_2\} \neq \emptyset$, $|I| = r + 1 + |L_1| + |L_2| + |V(G')|/2 = |V(G_1)| + r + 1$. Hence $\mathrm{miss}(G) = \{|V(G_1)| + r + 1, |V(G_1)| + 2r\}$ which implies $\mu_\alpha(G) = r - 1$.

Now, suppose that conditions of Case (ii) hold. Let z and z' be the two vertices of G_0. Thus there exist $a \in N_G(u_1) \cap V(G_1)$ and $b \in N_G(u_2) \cap V(G_1)$ such that z is adjacent to a and z' is

adjacent to b. Let I be an MIS of G. If $I \cap \{z, z'\} = \emptyset$, $\{a, b\} \subseteq I$, $I \cap \{u_1, u_2\} = \emptyset$, and $F_1 \cup F_2 \subseteq I$, which implies $|I| = |V(G_1)| + 2r$. If $I \cap \{z, z'\} \neq \emptyset$, there are two possibilities: $I \cap \{u_1, u_2\} = \emptyset$ and then $F_1 \cup F_2 \subseteq I$, which implies that $|I| = |V(G_1)| + 2r + 1$, and $I \cap \{u_1, u_2\} \neq \emptyset$, which implies that $|I| = |V(G_1)| + r + 2$. Hence $\text{miss}(G) = \{|V(G_1)| + r + 2, |V(G_1)| + 2r, |V(G_1)| + 2r + 1\}$ and we can conclude $\mu_\alpha(G) = r - 1$. □

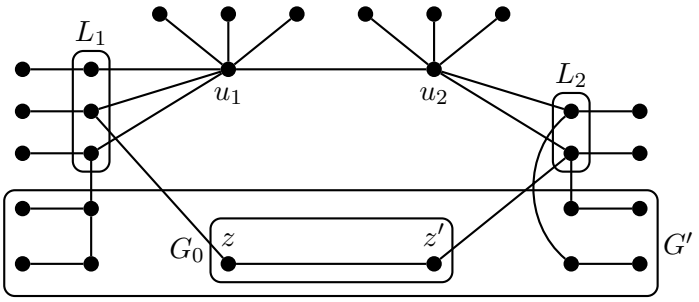


Figure 2: Graph G of girth 6 and two vertices of type 3. $G \in \mathcal{I}_3$ with $\text{miss}(G) = [14, 16]$.

Since the conditions of Theorem 2 can be tested in polynomial time, we have an efficient algorithm to recognize connected graphs G of girth at least 6, with exactly two type r vertices having $\mu_\alpha(G) = r - 1$.

Corollary 1. *Given a connected graph G of girth at least 6 and with exactly two type r vertices, it can be decided in polynomial time whether $\mu_\alpha(G) = r - 1$.*

Barbosa and Hartnell [1998] show sufficient conditions for a graph to belong to $\mathcal{M}_2$ or $\mathcal{I}_2$. They show that for any r, positive integer, the set $\mathcal{I}_r$ is not empty, since given any graph G of order n, it is possible to obtain G' from G by adding two leaves to each vertex of G. If $\alpha(G) = t$, then G' has MIS of sizes $2(n - t), 2(n - (t - 1)), \ldots, 2n$, and, therefore, G' belongs to $\mathcal{I}_{2(t-1)}$.

Theorem 2 has some consequences to $\mathcal{I}_r$. Observe that the graphs G considered have at least two and at most three distinct sizes of MIS and they are consecutive in the following conditions: if $r = 2$ and the girth of G is at least 6, as proved in Ekim et al. [2018]; and if $r = 3$, the girth of G is exactly 6, and $V(G_0) \neq \emptyset$. In the following cases the sizes of MIS of G are not consecutive: if $r \geq 3$ and the girth of G is at least 7, and if $r \geq 4$ and the girth of G is at least 6. We summarize these conditions in Corollary 2;

Corollary 2. *Let $r \geq 3$ and let G be a graph of girth at least 6 with $\mu_\alpha(G) = r - 1$ such that G contains exactly two vertices of type r. Then, $G \in \mathcal{I}_r$ only if $r = 3$ and the girth of G is exactly 6.*

Proof. Suppose $G \in \mathcal{I}_r$. Toward a contradiction, suppose first $r \geq 4$. By Theorem 2, G has at most 3 distinct sizes of MIS, which contradicts that $G \in \mathcal{I}_r$. Now, suppose that the girth of G is at least 7. If $V(G_0) \neq \emptyset$, by Theorem 2, Case (ii), G has girth exactly 6. If $V(G_0) = \emptyset$, by Case (i), G has at most 2 distinct sizes of MIS. However, since $r \geq 3$, this contradicts that $G \in \mathcal{I}_r$. □

Next, we consider $r \geq 3$ and G is a graph of girth at least 6 with exactly one vertex of type r such that $\mu_\alpha(G) = r - 1$. We bound the number of vertices of type $r - 1$ in G.

Lemma 7. *Let $r \geq 3$ and let G be a graph of girth at least 6 with exactly one vertex u_1 of type r. If $\mu_\alpha(G) = r - 1$, any vertex of G of type $r - 1$ is adjacent to u_1.*

Proof. Let G be a graph having $\mu_\alpha(G) = r-1$ with a vertex u_1 that is a vertex of type r. Suppose that there is a vertex u_2 of type $r-1$, which is not adjacent to u_1. Let F_i be the set of leaves of u_i, $i = 1, 2$ and let I be an MIS of G that contains $\{u_1, u_2\}$. Let $I' = (I - \{u_1, u_2\}) \cup F_1 \cup F_2$. Extend I' to a maximum independent set of G. It follows that $|I'| - |I| \geq |F_1| + |F_2| - 2 = 2r - 3 \geq r$, which contradicts that $\mu_\alpha(G) = r - 1$. Hence u_1 and u_2 are adjacent. □

Proposition 1. *Let G be a graph of girth at least 6 with $\mu_\alpha(G) = r-1$ and exactly one vertex u_1 of type r. If $r = 3$, then G has at most two vertices of type 2; and if $r \geq 4$, then G has at most one vertex of type $r-1$.*

Proof. Let G be a graph having $\mu_\alpha(G) = r - 1$ with a vertex u_1 that is a vertex of type r. First suppose $r = 3$ and there are three vertices in G, say u_2, u_3 and u_4, which are of type 2. By Lemma 7 each one of these vertices must be adjacent to u_1. By the girth restriction, vertices u_2, u_3 and u_4 are independent. Let F_i be the set of leaves of u_i, $i = 1, 2, 3, 4$ and let I be an MIS of T that contains $\{u_2, u_3, u_4\}$. Let $I' = I - (\{u_2, u_3, u_4\}) \cup F_2 \cup F_3 \cup F_4$. Extend I' to a maximum independent set of T. It is clear that $|I'| - |I| \geq |F_2| + |F_3| + |F_4| - 3 = 3$. This contradicts $\mu_\alpha(G) = 2$.

Now, suppose $r \geq 4$ and there are two vertices in G, say u_2 and u_3, which are of type $r-1$. By Lemma 7, each one of these vertices must be adjacent to u_1. By the girth restriction, vertices u_2 and u_3 are independent. Let F_i be the set of leaves of u_i, $i = 1, 2, 3$, and let I be an MIS of T that contains $\{u_2, u_3\}$. Let $I' = I - \{u_2, u_3\} \cup (F_2 \cup F_3)$. Extend I' to a maximum independent set of T. It is clear that $|I'| - |I| \geq |F_2| + |F_3| - 2 = 2r - 4 \geq r$, since $r \geq 4$. This contradicts $\mu_\alpha(G) = r - 1$. Therefore, G has at most one vertex of type $r-1$. □

Proposition 2 provides a procedure to build graphs that belong to $\mathcal{I}_r$ from other graphs of smaller order that also belong to $\mathcal{I}_r$. We call a vertex v in a graph G of vertex type-A if $i(G - v) \geq i(G)$, and type-B if $i(G - v) = i(G) - 1$.

Proposition 2. *Let $G \in \mathcal{I}_r$ for some $r \geq 2$ and H a graph that is a copy of K_2. Let G' be the graph obtained from G and H by connecting a vertex of H by an edge to a vertex v of G. (i) If v is type-A, then $G' \in \mathcal{I}_r$; and (ii) if v is type-B, then $G' \in \mathcal{I}_{r+1}$.*

Proof. Let G and H be graphs that satisfy the described hypotheses. Since $G \in I_r$, we may assume that $\mathrm{miss}(G) = [s, s+r-1]$, for some s, positive integer. Consider the graph H with vertices x and y. Let v be a vertex of type-A for Case (i) or a vertex of type-B for Case (ii). The graph H must be connected to G by an edge between v to x to obtain G'. Let I be any MIS of G. Note that it is always possible to extend I to an MIS I' de G' by adding the vertex y. Hence $[s+1, s+r] \subseteq \mathrm{miss}(G')$. Since it was added to G only two adjacent vertices, there is not in G' an MIS of size greater than $s+r$. For Case (i), we show that G' does not have an MIS of size less than $s+1$. A smallest MIS of G has size s and, by hypothesis, v is a vertex of type-A. Hence a smallest MIS of $G - v$, say J, has size at least s. If J is extended to an MIS J' of G' that has vertex x, we conclude that $|J'| \geq s+1$. Therefore, $\mathrm{miss}(G') = [s+1, s+r]$ and $G' \in I_r$. For Case (ii), a smallest MIS of G has size s and, by hypothesis, v is a vertex of type-B. So, there exists a smallest MIS of $G - v$, say J, which has size $s-1$. If J is extended to an MIS J' of G' that has vertex x, then we conclude that $|J'| = s$. Hence $\mathrm{miss}(G') = [s, s+r]$ and, therefore, $G' \in I_{r+1}$. □

4. Results on trees

Staples [1975] and Ravindra [1977] independently characterized the well-covered bipartite graphs, which includes all trees. A tree is well-covered if and only if all its pendant edges form

a perfect matching. Therefore, in a well-covered tree, each vertex is a support vertex with exactly one leaf or a leaf. The trees in $\mathcal{M}_2$ were characterized by Finbow et al. [1994].

It was presented in Proposition 2 a procedure to build graphs in $\mathcal{I}_r$ from other graphs having smaller order that also belong to $\mathcal{I}_r$. This result can be applied as part of a possible constructive characterization of trees in $\mathcal{I}_r$. If G is a tree in $\mathcal{I}_r$, for some $r \geq 2$, the resulting graph obtained from the described operation is also a tree with two additional vertices. Depending on the chosen vertex v, the new tree belongs to $\mathcal{I}_r$ or to $\mathcal{I}_{r+1}$. For instance, there are 32 distinct trees in $\mathcal{I}_3$ with 10 vertices. Of these trees, 30 can be obtained from the described operation in Proposition 2 from trees that belong to $\mathcal{I}_2 \cup \mathcal{I}_3$ with 8 vertices.

If G is a tree and $G \in \mathcal{I}_r$, for $r \geq 3$, then by Corollary 2 G has at most one vertex of type r. Next, we present a result related to the trees in $\mathcal{I}_3$ with exactly one type 3 vertex and exactly two type 2 vertices.

Theorem 3. *Let T be a tree with exactly one vertex u_1 of type 3, exactly two vertices u_2 and u_3 of type 2, and with no type k vertices for $k \geq 4$. Then $T \in \mathcal{I}_3$ if and only if $\{u_2, u_3\} \subseteq N_T(u_1)$ and any other vertex of T is of type 1 or a leaf.*

Proof. Let T be a tree as described. First, suppose $T \in \mathcal{I}_3$. Hence $\mu_\alpha(T) = 2$. By Lemma 7, $\{u_2, u_3\} \subseteq N_T(u_1)$. By Lemma 5, u_1 does not have a neighbor that is an internal vertex with no leaves. Now, we show that the neighbors of u_2 and u_3, if they exist, are support vertices. Toward a contradiction, suppose that v is a vertex with no leaves adjacent to u_2. By symmetry, there is not loss of generality. Let I be the set of vertices that are, at the same time, at distance 3 from u_2 and distance 2 from v. By the girth restriction, I is independent. Now, consider the graph $T' = T - N_T[I]$. Observe that in T' the vertex v is a leaf of u_2 and u_2 has now 3 leaves. However, since T' has two vertices of type 3, by Corollary 2 $T' \notin \mathcal{I}_3$ and, by Lemma 1 $\mu_\alpha(T) \geq 3$, contradicting that $T \in \mathcal{I}_3$. Therefore, all vertices adjacent to u_2 and to u_3 are vertices of type 1.

Let $L_i = N_T(u_i) - (\{u_1, u_2, u_3\} \cup F_i)$, for $i = 1, 2, 3$. By the girth restriction, L_i is independent. Moreover, all vertices in L_i are of type 1. See Figure 3 for an illustration. Let L'_i be the set of all leaves of vertices in L_i, for $i = 1, 2, 3$. Note that $|L_i| = |L'_i|$. Now, let $I = \bigcup_{i \in \{1,2,3\}} (F_i \cup L'_i)$ and let $T' = T - N_T[I]$. We claim that T' is well-covered. Observe that $\alpha(T) = \alpha(T') + |I| = \alpha(T') + 7$ and $i(T) = i(T') + 5$. If T' is not well-covered, $i(T') < \alpha(T')$, and then $\alpha(T) - i(T) \geq 3$. By Theorem 1, each component of T' is isomorphic to K_1 or to a graph with a perfect matching formed by its pendant edges. Since T is has no cycles, it is easy to see that no component of T' is isomorphic to K_1. Therefore, T' has a perfect matching formed by its pendant edges.

By using similar arguments to the ones used in Claim 3 in the proof of Theorem 2, we can conclude that if T' has a component with only vertices that do not have leaves in T, then this component is isomorphic to K_2. However, in this case, one of the vertices of K_2 would have a neighbor in L_j and the other in $L_{j'}$, where $j \neq j'$ and $j, j' \in \{1, 2, 3\}$. Hence they are under a 6-cycle containing vertex u_1 and at least one of u_2 and u_3, or under a 7-cycle containing u_1, u_2 and u_3. Therefore, $V(T)$ contains exactly the vertices u_1, u_2 and u_3, their leaves ($F_1 \cup F_2 \cup F_3$), and any other vertex of T is of type 1 or a leaf.

Conversely, let T be a tree with exactly one vertex u_1 of type 3 and exactly two vertices, u_2 and u_3, of type 2, such that u_2 and u_3 are adjacent to u_1 and any other vertex of T is of type 1 or a leaf. Let I be an MIS of T and $s = |L_1| + |L_2| + |L_3| + |V(T')|/2$. If $I \cap \{u_1, u_2, u_3\} = \emptyset$, I contains the leaves in $F_1 \cup F_2 \cup F_3$ and $|I| = s + 7$. If $I \cap \{u_1, u_2, u_3\} = \{u_1\}$, I contains the leaves in $F_2 \cup F_3$ and $|I| = s + 5$. If $I \cap \{u_1, u_2, u_3\} = \{u_2, u_3\}$, I contains the leaves in F_1 and

$|I| = s+5$. If $I \cap \{u_1, u_2, u_3\} = \{u_2\}$ or $I \cap \{u_1, u_2, u_3\} = \{u_3\}$, I contains the leaves in F_1 and the leaves in F_2 or the ones in F_3. In both cases, $|I| = s+6$. Therefore, $\text{miss}(T) = [s+5, s+7]$, which implies that $T \in \mathcal{I}_3$. □

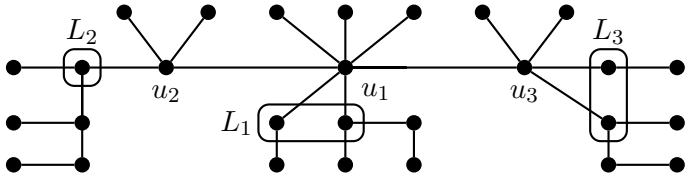


Figure 3: Tree T with exactly one vertex of type 3 and exactly two vertices of type 2. $T \in \mathcal{I}_3$ with $\text{miss}(T) = [14, 16]$.

In a tree T with n vertices, $\alpha(T) \leq n-1$, and since a tree is a bipartite graph, $\alpha(T) \geq \lceil \frac{n}{2} \rceil$. The upper bound is achieved by the complete bipartite graph $K_{1,n-1}$, also known by star graph. This graph has $i(T) = 1$. Hence for each $n \geq 2$, there is only one tree with $i(T) - 1$ and $\alpha(T) = n - 1$. Note that this graph has $\mu_\alpha(T) = n - 2$, which is the maximum possible for any connected graph. Favaron [1992] shows that in a tree T with n vertices and ℓ leaves $i(T) \leq \frac{(n+\ell)}{3}$. For instance, the paths P_{3k+1} when $k \geq 1$ achieve this bound. In the following results, we show the number of non-isomorphic trees having specific sizes of MIS and prove that there are exactly $\lceil \frac{n}{2} \rceil - 1$ non-isomorphic trees T with n vertices having $\mu_\alpha(T) = n - 4$.

Theorem 4. *For $n \geq 3$, there are exactly $n-3$ non-isomorphic trees T on n vertices with $\alpha(T) = n-2$.*

Proof. Let T be a tree with n vertices, $I \subseteq V(T)$ be a maximum independent set of T, and $S = V(T) - I$. Since $\alpha(T) = n - 2$, $|S| = 2$. Let $S = \{u, v\}$. Thus there are two possible cases: $uv \in E(T)$ and $uv \notin E(T)$.

Case 1: $uv \in E(T)$. Each vertex $x \in I$ has $d_T(x) = 1$, otherwise the 3-cycle $\langle x, u, v \rangle$ is formed. Thus, since $uv \in E(T)$, it is clear that $d_T(u) + d_T(v) = n$. Therefore, the following combinations of $d_T(u)$ and $d_T(v)$ are obtained: $(n-2, 2), (n-3, 3), \ldots, (\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$. That is, there are exactly $\lfloor \frac{n}{2} \rfloor - 1$ values for $d_T(u)$ and $d_T(v)$, with $d_T(u) + d_T(v) = n$, that form only non-isomorphic trees.

Case 2: $uv \notin E(T)$. Since T is connected and, in this case, $uv \notin E(T)$, it is easy to see that $\exists x \in I$ such that $ux, xv \in E(T)$. Suppose $\exists y \in V(T)$ with $y \neq x$, such that $uy, vy \in E(T)$. in this case the 4-cycle $\langle u, x, v, y \rangle$ is obtained. Thus there is only one vertex $x \in I$ such that $d_T(x) > 1$ and, since $|S| = 2$, we have $d_T(x) = 2$. Given that all the edges of T have an endpoint in S and $uv \notin E(T)$, it follows that $d_T(u) + d_T(v) = n - 1$. Therefore, analogously to Case 1, one can combine the values of $d_T(u)$ and $d_T(v)$, with $d_T(u) + d_T(v) = n - 1$, forming $\lceil \frac{n}{2} \rceil - 1$ non-isomorphic trees. The combinations are the following: $(n-2, 1), (n-3, 2), \ldots, (\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil - 1)$.

Thus, putting together all the possible combinations of $d_T(u)$ and $d_T(v)$ mentioned in the two cases shown above, $\lfloor \frac{n}{2} \rfloor - 1$ and $\lceil \frac{n}{2} \rceil - 1$, we obtain $n - 2$ trees T on n vertices with $\alpha(T) = n-2$. However, there are two trees, say T_1 and T_2, that satisfy Cases 1 and 2, respectively, such that $T_1 \cong T_2$. Suppose that in T_1, vertices $u_1, v_1 \in S_1$, with $d_T(u_1) = n - 2$ and $d_T(v_1) = 2$. In T_2, let $u_2, v_2 \in S_2$ with $d_T(u_2) = n - 2$ and $d_T(v_2) = 1$. It is easy to see that T_1 and T_2 are isomorphic trees, and, by the structure of the described trees, there is no other isomorphism. Therefore, there exist $n - 3$ non-isomorphic trees T on n vertices with $\alpha(T) = n - 2$. □

Theorem 5. *For $n \geq 3$, there are exactly $n-3$ non-isomorphic trees T on n vertices with $i(T) = 2$.*

Proof. Let T be a tree with n vertices, $I \subseteq V(T)$ be an MIS of T with $|I| = 2$, and $S = V(T) - I$. Given that $i(T) = 2$, we may assume that $I = \{u, v\}$. Clearly $|S| = n - 2$. We consider two possible cases: $\exists x \in S$ such that $ux, vx \in E(T)$, and $\nexists x \in S$ such that $ux, vx \in E(T)$.

Case 1: $\exists x \in S$ such that $ux, vx \in E(T)$. If $\exists y \in S$, with $y \neq x$, such that $uy, vy \in E(T)$, the 4-cycle $\langle u, x, v, y\rangle$ is obtained, which is impossible, since T is a tree. Therefore, there is only one vertex $x \in S$ such that $ux, vx \in E(T)$ and, consequently, $d_T(x) \geq 2$. Suppose $\exists y \in S$, with $y \neq x$, such that $yx \in E(T)$. Since I is a dominating set, we may assume $uy \in E(T)$, without loss of generality. However, that is impossible, as the 3-cycle $\langle u, y, x\rangle$ is formed. Since $d_T(x) \geq 2$ and $\nexists y \in S$ with $yx \in E(T)$, it follows that $d_T(x) = 2$.

It is easy to see that S is an independent set and $\forall y \in S$, with $y \neq x$, $d_T(y) = 1$. Hence we have that $d_T(u) + d_T(v) = n - 1$, as Case 2 from Theorem 4. Therefore, there exist exactly $\lceil \frac{n}{2} \rceil - 1$ combinations of the values of $d_T(u)$ and $d_T(v)$, with $d_T(u) + d_T(v) = n - 1$, that form only non-isomorphic trees.

Case 2: $\nexists x \in S$ such that $ux, vx \in E(T)$. Let $x \in S$ such that $ux \in E(T)$ and $vx \notin E(T)$. Hence we have that $\exists y \in S$ such that $xy \in E(T)$. If $yu \in E(T)$, the 3-cycle $\langle u, x, y\rangle$ is obtained. Thus if $ux \in E(T)$, necessarily $vy \in E(T)$, since I is a dominating set. Suppose $\exists z \in S$, with z distinct from x and y. By symmetry, suppose $uz \in E(T)$. If $xz \in E(T)$, it has the 3-cycle $\langle u, x, z\rangle$. If $yz \in E(T)$, it is formed the 4-cycle $\langle u, z, y, x\rangle$. We can conclude that $\forall z \in S$, with z distinct from x and y, $xz, yz \notin E(T)$.

Consider that $\exists z, w \in S$, with z, w, x and y all distinct to each other, such that $zw \in E(T)$. Analogously to the case of x and y above, we have that $zu, wy \in E(T)$, without loss of generality. However, the 5-cycle $\langle u, x, y, w, z\rangle$ would be formed. Hence the graph induced by S has only the edge xy and $d_T(x) = d_T(y) = 2$. Moreover, $\forall z \in S$, with $z \neq x$ and $z \neq y$, $d_T(z) = 1$. Since every edge in $E(T) - \{xy\}$ has one endpoint in I, it follows that $d_T(u) + d_T(v) = n - 2$. Thus we can obtain $\lfloor \frac{n}{2} \rfloor - 1$ non-isomorphic trees with the following combinations of values of $d_T(u)$ and $d_T(v)$: $(n-3, 1), (n-4, 2), \ldots, (\lceil \frac{n}{2} \rceil - 1, \lfloor \frac{n}{2} \rfloor - 1)$.

Thus, putting together all possible combinations of $d_T(u)$ and $d_T(v)$ mentioned in the two cases above, $\lfloor \frac{n}{2} \rfloor - 1$ and $\lceil \frac{n}{2} \rceil - 1$, respectively, we can conclude that there are $n - 2$ trees T on n vertices with $i(T) = n - 2$. However, there are two trees, say T_1 and T_2, that satisfy Cases 1 and 2, respectively, such that $T_1 \cong T_2$. Suppose that in T_1, vertices $u_1, v_1 \in I_1$, with $d_T(u_1) = n - 3$ and $d_T(v_1) = 2$. In T_2, let $u_2, v_2 \in S_2$ with $d_T(u_2) = n - 3$ and $d_T(v_2) = 1$. It is easy to verify that T_1 and T_2 are isomorphic trees and, by the structure of the described trees, there is no other isomorphism. Therefore, we can conclude that there exist $n - 3$ non-isomorphic trees T on n vertices with $i(T) = 2$. □

Theorem 6. *For $n \geq 3$, there are exactly $\lceil \frac{n}{2} \rceil - 1$ non-isomorphic trees T on n vertices with $i(T) = 2$ and $\alpha(T) = n - 2$, that is, $\mu_\alpha(T) = n - 4$.*

Proof. Let T be a tree on n vertices with $i(T) = 2$ and $\alpha(T) = n - 2$. Since T is a bipartite graph, there is a partition X, Y of T where $|X| = 2$, $|Y| = n - 2$, and the vertices in a same partition are independent. Hence T is a tree that satisfies both, Case 2 of Theorem 4, and Case 1 of Theorem 5. Therefore, there are exactly $\lceil \frac{n}{2} \rceil - 1$ non-isomorphic trees T on n vertices with $i(T) = 2$ and $\alpha(T) = n - 2$, that is, $\mu_\alpha(T) = n - 4$. □

References

Barbosa, R., Cappelle, M. R., and Rautenbach, D. (2013). On graphs with maximal independent sets of few sizes, minimum degree at least 2, and girth at least 7. *Discrete Mathematics*, 313(16):1630 – 1635.

Barbosa, R. and Hartnell, B. (1998). Some problems based on the relative sizes of the maximal independent sets in a graph. *Congressus Numerantium*, pages 115–121.

Bondy, J. A. and Murty, U. S. R. (1976). *Graph Theory with Applications*. Elsevier, New York.

Cappelle, M. R. and Rautenbach, D. (2013). Extending berge's and favaron's results about well-covered graphs. *Discrete Mathematics*, 313(23):2742 – 2747.

Caro, Y., Sebő, A., and Tarsi, M. (1996). Recognizing greedy structures. *Journal of Algorithms*, 20(1):137 – 156.

Chvátal, V. and Slater, P. J. (1993). A note on well-covered graphs. *Annals of Discrete Mathematics*, 55:179 – 181.

Ekim, T., Gözüpek, D., Hujdurović, A., and Milanič, M. (2018). On Almost Well-Covered Graphs of Girth at Least 6. *Discrete Mathematics & Theoretical Computer Science*, vol. 20 no. 2.

Ekim, T., Gözüpek, D., Hujdurović, A., and Milanič, M. (2020). Mind the independence gap. *Discrete Mathematics*, 343(9):111943.

Favaron, O. (1992). A bound on the independent domination number of a tree. *Vishwa International Journal of Graph Theory*, 1:19 – 27.

Finbow, A. and Hartnell, B. (1983). A game related to covering by stars. *Ars Combinatoria*, 16:189–198.

Finbow, A., Hartnell, B., and Nowakowski, R. J. (1993). A characterization of well-covered graphs of girth 5 or greater. *J. Combin. Theory, Ser. B*, 57:44–68.

Finbow, A., Hartnell, B., and Whitehead, C. (1994). A characterization of graphs of girth eight or more with exactly two sizes of maximal independent sets. *Discrete Mathematics*, 125(1):153 – 167.

Hartnell, B. L. (1999). Well-covered graphs. *J. Combin. Math. Combin. Comput.*, 29:107–115.

Hartnell, B. L. and Rall, D. (2013). On graphs having maximal independent sets of exactly t distinct cardinalities. *Graphs and Combinatorics*, 29:519–525.

Plummer, M. D. (1970). Well-covered graphs. *J. Combin. Theory*, 8:91–98.

Plummer, M. D. (1993). Well-covered graphs: a survey. *Quaestiones Math.*, 16:253–287.

Ravindra, G. (1977). Well-covered graphs. *J. Combin. Inform. System Sci*, 2:20–21.

Sankaranarayana, R. S. and Stewart, L. K. (1997). Complexity results for well-covered graphs. *J. Combin. Inform. System Sci*.

Staples, J. (1975). *On Some Subclasses of Well-Covered Graphs*. PhD thesis, Vanderbilt University.

Galoá { Este trabalho foi publicado utilizando o **Galoá** proceedings