

Partitioning split graphs into two trees

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RESUMO

Seja $G = (V, E)$ um grafo simples e não direcionado. Neste trabalho, consideramos o problema de particionar $V(G) = A \cup B$, $A \cap B = \emptyset$, tal que cada A e B induzem uma árvore (i.e., um subgrafo acíclico e conexo). Os grafos que admitem essa partição foram denominados *grafos de Yutsis* e seus estudos estão relacionados aos estudos de grafos planares Hamiltonianos. O reconhecimento de grafos de Yutsis é NP-completo para grafos gerais e, até o momento, não há estudos considerando a caracterização de grafos de Yutsis para classes de grafos específicas. Este trabalho se restringe a reconhecer grafos split (grafos cujo conjunto de vértices pode ser particionado em um conjunto independente e num clique) que são de Yutsis. Mais especificamente, fornecemos uma caracterização por subgrafos proibidos que conduz a um algoritmo polinomial de reconhecimento dos grafos split que são Yutsis.

PALAVRAS CHAVE. Partição em grafos, grafos de Yutsis, grafos Split.

Tópicos: Teoria dos grafos, problemas de partição, caracterizações

ABSTRACT

Let $G = (V, E)$ be a simple, undirected graph. In this work, we consider the problem of partitioning $V(G) = A \cup B$, $A \cap B = \emptyset$, such that each A and B induces a tree (i.e., an acyclic and

connected subgraph). Graphs which admit such partition were called *Yutsis graphs* and its studies are related to hamiltonian planar graph studies. Yutsis graph recognition is NP-complete for general graphs and there are not studies considering the characterizations of Yutsis graphs for specific graph classes, so far. Our work is restricted to recognizing split graphs (graphs whose set of vertices can be partitioned into an independent set and in a clique). More specifically, we present a characterization by forbidden subgraphs that provides a polynomial algorithm for recognizing split graphs that are Yutsis.

KEYWORDS. Graph partition problems, Yutsis graphs, Split graphs,

Paper topics: Graph theory, partition problems, characterizations

1. Introduction

Several graph problems can be viewed as partition problems, such as, vertex coloring (Mala guti e Toth [2010]), edges coloring (Holyer [1981]), recognition of (k, ℓ) -graph (Brandstädt [1996]; Bravo et al. [2011]; Couto et al. [2018]), tessellability problem (Abreu et al. [2020]), and others. These problems have applications in graph theory and in many other fields, such as frequency assignments (Aardal et al. [2007]), or quantum computation (Portugal et al. [2016]). Another problem with applications in quantum theory is the recognition of the Yutsis graphs (Van Dyck e Fack [2007]), which aims to decide if a graph has a vertex partition into two trees.

Let $G = (V, E)$ be a simple and undirected graph. A cut $[S, T]$ of G is a partition of V into two subsets S and $T = V \setminus S$. The cut-set $\partial(S)$ of a cut $[S, T]$ is the set of edges that have one endpoint in S and the other endpoint in T . A minimal nonempty edge cut of G is called a bond. Since every nonempty bond determines a cut $[S, T]$ such that $G[S]$ and $G[T]$ are both connected, every bond of G has size at most $|E(G)| - |V(G)| + 2$. A graph G has a bond of size $|E(G)| - |V(G)| + 2$ if and only if $V(G)$ can be partitioned into two parts such that each part induces a tree. These are the *Yutsis graphs* (Duarte et al. [2019]).

If a Yutsis graph is a cubic graph, i.e. a 3-regular graph, the partition of the vertex set from the largest bond results in two sets of equal size (Aldred et al. [2009]). Cubic Yutsis graphs were introduced by Jaeger [1974] as cubic dual Hamiltonian graphs. These graphs are studied in the context of the quantum theory of angular momentum field, where they are used to generate summation formula for general recoupling coefficients (Van Dyck e Fack [2007]).

A folklore theorem states that if G is a connected planar graph, then a set of edges is a cycle in G if and only if it corresponds to a bond in the dual graph of G , see Godsil e Royle [2013]. Therefore, a planar graph G has a bond of size $|E(G)| - |V(G)| + 2$ (which is the maximum possible) if and only if its dual is a planar Hamiltonian graph. In other words, the class of planar Yutsis graphs is exactly the class of dual Hamiltonian graphs. Thus, computing partitions of the vertices of planar graphs into two trees is equivalent to computing Hamiltonian cycles of planar graphs. Garey et al. [1976] proved that the problem of establishing whether a planar graph is Hamiltonian is NP-complete, thus, the problem of determining whether a planar graph is partitionable into two trees is also NP-complete, i.e., the recognition of Yutsis graphs is NP-complete even restricted to planar graphs.

Besides, the problem of deciding whether a given cubic graph is a Yutsis graph is also NP-complete, and studies regarding parameterized complexity were developed in Duarte et al. [2019]. In spite of that, there are not studies considering the characterizations of Yutsis graphs for specific graph classes, so far. Based on that, we consider the problem of determining if a split graph G is a Yutsis graph, i.e., determine if G can be partitioned into two trees.

This paper is organized as follows: in Section 2, we present preliminary concepts, results and notation; in Section 3, we present characterizations in terms of forbidden subgraphs and characterization that yields to a polynomial-time algorithm for split Yutsis graphs; in Section 4 we conclude the paper with final remarks and further work.

2. Preliminaries

Throughout this paper, a graph $G = (V(G), E(G))$ is considered to be connected and finite. The set $N_G(v) = \{u \mid uv \in E(G)\}$ is the *neighborhood* of the vertex v in V and the *degree* of v is $d(v) = |N_G(v)|$. When the context does not present ambiguous, we denote the neighborhood of v in G as $N(v)$. A *pendant* vertex is a vertex v such that $d(v) = 1$. A subset S of $V(G)$ is an *independent set* if no two vertices of S are adjacent in G . A *clique* is a subset of vertices of a graph

such that every two distinct vertices in the clique are adjacent. A graph $G = (V, E)$ is a *split graph* if its vertex set V can be partitioned into an independent set and a clique.

A k -clique is a clique with k vertices and $\omega(G)$ is the length of its maximum clique of G . A *trivial graph* is a graph with a unique vertex. A *tree* is a connected acyclic undirected graph. A *double star graph* is a tree formed by a path $P_2 = u, v$ so that there are m adjacent vertices to u and n adjacent vertices to v . A graph $G = (V(G), E(G))$ is a *bipartite graph* if $V(G)$ can be partitioned into two sets X and Y , such that each edge of $E(G)$ has one endpoint in X and another one in Y . Hence, there is no edge formed between two vertices of a same set. Therefore, bipartite graphs do not have odd cycles as induced subgraphs.

A *proper coloring* of a graph is a labeling of the vertices, such that two adjacent vertices do not have the same color. A graph G is k -colorable if there is a proper coloring of G with at most k colors.

The smallest integer k such that G is k -colorable is the *chromatic number* of G , denoted by $\chi(G)$. Two graphs G and H with a same vertex set $V_n = \{1, 2, \dots, n\}$ are said to be *isomorphic* if there is a permutation p of V_n such that u, v is an edge of $E(G)$ if and only if $p(u), p(v)$ is an edge of $E(H)$.

For other definitions in graphs see Bondy e Murty [1976].

Yutis graphs are the graphs that can be partitioned into two parts such that each part induces a tree. Hence, we say that a graph G has a *two-tree partition* if and only if G is a Yutis graph. Fig. 1 depicts a graph that has a two-tree partition.

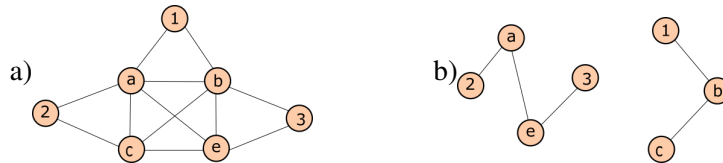


Figure 1: (a) A graph G . (b) A two-tree partition of G .

Lemma 1. If a graph G has a two-tree partition, then $\chi(G) \leq 4$.

Proof. Suppose that G has a two-tree partition, and let T_1 and T_2 be such two trees. Since T_1 and T_2 are bipartite graphs, $\chi(T_1), \chi(T_2) \leq 2$. Therefore, $\chi(G) \leq \chi(T_1) + \chi(T_2) \leq 4$. $\square$

Corollary 1. If a graph G contains a K_5 as a subgraph then, G does not have a two-tree partition.

Proof. It follows from Lemma 1 and the fact that a K_5 is not 4-colorable. $\square$

3. Two-tree partition of split graphs

In this section, we provide characterizations of split graphs that have two-tree partitions. First, we present cases that split graphs that do not have a two-tree partition, and we are able to prove that having two-tree partition is not a hereditary property for split graphs (Corollary 2). A *n-sun graph* is graph with $2n$ vertices consisting of a central complete graph K_n with an outer ring of n vertices, each of which is joined to both endpoints of the closest outer edge of the central core. We denote by S_3 the 3-sun graph (Fig. 2).

Lemma 2. A graph isomorphic to a S_3 does not have a two-tree partition.

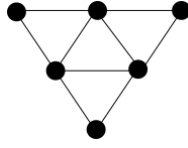


Figure 2: 3-sun graph.

Proof. Let $G = S_3$. Suppose G admits such a partition into T_1 and T_2 . We are going to label vertices of G by labels 1 or 2, according to the partition it belongs to (either to T_1 or T_2 , respectively). One can observe, without loss of generality, that the central complete graph K_3 of G must be labeled in such a way that two of its vertices are labeled by 1 and another one is labeled by 2. Suppose that from the remaining vertices (say a, b, c), vertices a and b are labeled by 1 and vertex c is labeled by 2. It is easy to see that no matter the way we label these remaining vertices, either we get a cycle or we contradict the connectivity property. $\square$

Corollary 2. *For any split graph, the property of having two-tree partition is not hereditary.*

Proof. By Lemma 2, it follows that a graph isomorphic to a S_3 cannot have its vertices partitioned into two-trees. Now, let H be the graph presented in Fig. 3(a), it is easy to see that H has a S_3 as induced subgraph. However, as illustrated in Fig. 3(b) the graph H can be partitioned into two-trees. $\square$

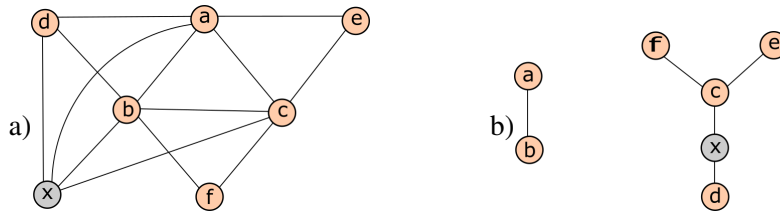


Figure 3: a) A graph that has a S_3 as an induced subgraph. b) Solution of a two-tree partition (T_1, T_2) , such that $V(T_1) = \{a, b\}$, and $V(T_2) = \{c, d, e, f, x\}$.

Next, in Definition 1, we describe good partitions of the clique part of split graphs. The existence of a good partition is a necessary and sufficient condition for split graphs to have two-tree partitions, as proved in Theorem 1. Moreover, in Theorem 2 we determine a characterization for split graphs that have two-tree partitions in terms of forbidden subgraphs.

Definition 1. Let $G = (K, S)$ be a split graph. A good partition of K is a partition of K into two-trees T_1 and T_2 , such that for every vertex $v \in S$, $|N_{T_1}(v)| = 1$ or $|N_{T_2}(v)| = 1$. We denote such a good partition of K as $K = (T_1, T_2)$.

Lemma 3. Let $G = (K, S)$ be a split graph. If K admits a good partition, then G has a two-tree partition.

Proof. Let $K^* = (T_1, T_2)$ be a good partition of K . Since K^* is a clique, $|V(T_1)| < 3$ or $|V(T_2)| < 3$. For $|K^*| < 3$, the result follows trivially. Consider now the cases when $|K^*| = 3, 4$. Since G is connected, every vertex of S has at least one neighbor in $T_1 \cup T_2$. Moreover, since K^* has a good partition, for every $v \in S$, we have that $|N_{T_1}(v)| = 1$ or $|N_{T_2}(v)| = 1$. We construct a two-tree

partition (T_1^*, T_2^*) by including in T_1^* every vertex of T_1 and vertices of S with a unique neighbor in T_1 (cycles are not created and we keep connectivity). For T_2^* , we include every vertex of T_2 and vertices of $S \setminus T_1^*$ with a unique neighbor in T_2 (at this point, these vertices are pendant vertices of T_2 in the graph $G \setminus T_2^*$), and we keep connectivity and cycles are not created. $\square$

Theorem 1. *Let $G = (K, S)$ be a split graph. G has a two-tree partition if and only if G admits a good partition.*

Proof. Let $G = (K, S)$ be a split graph. Suppose G can be partitioned into two-trees T_1 and T_2 . Consider $T_1^* = V(T_1) \cap K$ and $T_2^* = V(T_2) \cap K$. Next, we show that every vertex $v \in S$ satisfies $|N_{T_1^*}(v)| < 2$ or $|N_{T_2^*}(v)| < 2$. Suppose there exists $v \in S$, such that $|N_{T_1^*}(v)| \geq 2$ and $|N_{T_2^*}(v)| \geq 2$. Without loss of generality, suppose $v \in T_1$. In this case, we get a contradiction since v would have two neighbors in T_1 inducing a cycle. Therefore, K admits a good partition. Suppose that K admits a good partition, then, by Lemma 3, G has a two-tree partition. $\square$

One can observe that a 3-sun (Fig. 2) does not have a good partition, therefore it does not have a two-tree partition. A S_4 graph has a two-tree partition, we can partition by clique by its two internal edges, thus all the vertices of the independent set will be adjacent to only one vertex of each part.

Theorem 1 leads us to a polynomial-time algorithm for recognizing split graphs that have two-tree partitions. For each clique, there is a constant number of eventual good partitions, because each clique has size at most four. Thus, it is necessary to verify for each vertex v of the independent set if v satisfies the conditions of the clique part to have a good partition.

Moreover, by Corollary 1, if a graph G has a two-tree partition then $\omega(G) \leq 4$. Hence, Theorem 2 characterizes split graphs that have two-tree partitions.

Theorem 2. *Let $G = (K, S)$ be a split graph and $\omega(G)$ the length of its maximum clique. G has a two-tree partition if and only if either:*

1. $\omega(G) \leq 2$ or;
2. $\omega(G) = 3$ and G is S_3 -free (see Fig. 2); or
3. $\omega(G) = 4$ and G is $\{H_1, H_2\}$ -free (see Fig. 4).

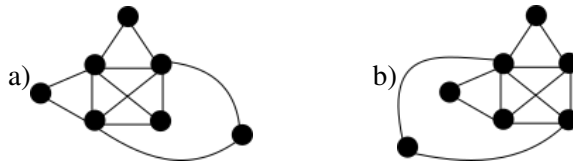


Figure 4: a) An example of a H_1 graph. b) An example of a H_2 graph.

Proof. $(\Leftarrow)$

1. Let G be a split graph with $\omega(G) \leq 2$. If $\omega(G) = 1$, then G has a two-tree partition (T_1, T_2) by considering, without loss of generality, T_1 containing only 1 vertex and T_2 empty. For

$\omega(G) = 2$, each clique is composed by a K_2 . If there is one more vertex in the graph, it must be connected to just one endpoint of the clique, otherwise there would be a K_3 in G , increasing $\omega(G)$. Therefore, G is a double-star. The clique part of G , which is a K_2 , is partitioned into two trees T_1 and T_2 such that each tree contains an endpoint of such a K_2 . Since the remaining vertices belong to the independent set, then such vertices are connected to either T_1 or T_2 , creating then, a two-tree partition of G .

2. Let G be a split S_3 -free graph with $\omega(G) = 3$. Hence, each vertex v of the independent set has $d(v) \leq 2$, otherwise, if one of the vertices of the independent set has $d(v) > 2$, there would be a K_4 in G , increasing $\omega(G)$. We can assume that, without loss of generality, $d(v) = 2$ for all v of the independent set, since any vertex v with $d(v) = 1$ does not interfere in the partition of the graph.

If the size of the independent set is at most 2, it is easy to see that there is a good partition. For graphs with the independent set greater than 2, we observe that since they are S_3 -free and $d(v) = 2$, for any combination of 3 vertices of the independent set, there is always an edge of the clique whose endpoints are not adjacent to a same vertex of the independent set. Thus, by the Lemma 3, the graph G admits a good partition. We assume that $|K| = 3$ and can be partitioned into T_1 and T_2 , so that two vertices are placed in T_1 and the other one is placed in T_2 . All vertices of the independent set that are adjacent to only one vertex of T_1 are added to that partition. The other vertices are added to T_2 .

3. Let G be a split graph $\{H_1, H_2\}$ -free with $\omega(G) = 4$. First we analyze the vertices v of the independent set that have $d(v) = 1$, they do not interfere in the partition, as they are added only to the adjacent vertex without creating cycles. We have that each vertex v of the independent set has $d(v) < 4$, otherwise, if there is a vertex of degree $d(v) = 4$, it implies $\omega(G) = 5$. For each vertex v of the independent set such that $d(v) = 3$, it is possible to add v to one of the trees T_1 or T_2 . Vertex v is adjacent to two vertices of one of the trees, but only one vertex of the other, so it can be added to it. In this way, we consider only the vertices of the independent set with degree equal to 2. In the clique K_4 there are three pairs of possible edges for a good partition. K can only be partitioned into two disjoint pairs of vertices, where one will be a subgraph of T_1 and the other of T_2 . Since G does not have H_1 nor H_2 , for each three vertices of the independent set, there is at least one pair of edges that can be part of the partition, forming a graph isomorphic to the one in Fig. 1, named H_3 . The graph H_3 has a good partition and by Lemma 3 it has two-tree partition.

($\Rightarrow$)

Let G be a graph that has a two-tree partition, then by Corollary 1, $\omega(G) \leq 4$.

1. Let G be a split graph, such that $\omega(G) \leq 2$. Note that if the graph has more than two vertices, to preserve connectivity the $\omega(G) \geq 2$, a contradiction with the fact that $\omega(G) = 1$, so the graph is trivial. Let G be a split graph that has a two-tree partition, with $\omega(G) = 2$ and it is not a double star. Since K_2 is a clique that has a good partition, by Lemma 3, the edges of the independent set cannot be adjacent to two vertices at the same time, and also, if there is a vertex adjacent to the two vertices, then $\omega(G) = 3$. Therefore, a split graph with $\omega(G) = 2$ can only have vertices of the independent set adjacent at one vertex of the clique, that is, a double star. A contradiction.

2. Let G be a split graph, with $\omega(G) = 3$ such that it has a two-tree partition and has a S_3 as an induced subgraph. Suppose that G has $|V(S_3)| + 1$ vertices. We denote $\{a, b, c\}$ the vertices of the clique and $\{d, e, f, g\}$ the vertices of the independent set. Without loss of generality, g is the vertex added to the S_3 . If g is adjacent to just one vertex of the clique or two vertices of the clique, g remains in the independent set. However this graph does not have a good partition, since at least one of the vertices of the independent set is adjacent just two vertices of the same tree, forming a cycle. If g is adjacent to just one vertex of the independent set, the graph is no longer a split graph. Moreover, if any edge is added between the vertices of S_3 , then we do not have a S_3 as an induced subgraph of G . Finally, the vertex g cannot be adjacent to all vertices of the clique, otherwise it would be $\omega(G) = 4$. Therefore, a contradiction. If G has a two-tree partition, with $\omega(g) = 3$, so G is a S_3 -free.

3. Let G a split graph and $\omega(G) = 4$, so by the Lemma 3, if G admits a good partition, then it has a two-tree partition. Suppose that G has a good partition and it has H_1 or H_2 as induced subgraph. For the complete subgraph K_4 there are three possible partition into two-trees: $\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}$.

Consider these pairs as those that do not share an endpoint in K_4 , where $\{e_1, e_2\}$ is the upper and lower edge pair, $\{e_3, e_3\}$, the edge pair on the sides and $\{e_5, e_6\}$ is an internal pair of edges.

$\{e_1, e_2\}$ is not a good partition. Either e_1 or e_2 has a vertex adjacent to exactly the one of the two edges e_1 or e_2 . Without loss of generality, suppose that it is e_1 . Hence, this vertex has to be placed in T_2 , but as it is not a neighbor of e_2 , T_2 is disconnected. Similarly, we have that $\{e_3, e_4\}$ and $\{e_5, e_6\}$ are not a good partition. Therefore, there is a vertex of the independent set that is adjacent to a vertex of just one of these edges. Hence, with out loss of generality, if that vertex is a neighbor of e_5 , then the subgraph is an H_1 , whereas if it is a neighbor of e_6 , then the subgraph is an H_2 .

□

4. Conclusion

In this work, we introduced the study of the partition of a graph into two trees, by considering the split graphs, which are the split Yutis graphs. We showed a characterization in terms of forbidden subgraphs for this class and proved that a split graph has a good partition if and only if it can be partitioned into two trees. Moreover, we observed that determining whether a split graph has a good partition is a polynomial-time solvable problem. Furthermore, we intend to study Yutis graphs for other graph classes, such as cographs, chordal graphs and (k, ℓ) -graphs.

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