

ON THE REDUCED RASTER POINTS FOR CUTTING AND PACKING PROBLEMS

Vinícius Loti de Lima and Thiago Alves de Queiroz

Institute of Mathematics and Technology, Federal University of Catalão,
Av. Dr. Lamartine P. Avelar, 1120, Setor Universitário, 75704-020, Catalão - GO, Brasil.
vini.lot@gmail.com, taq@ufg.br

RESUMO

Problemas de corte e empacotamento envolvendo itens de formato retangular podem ser resolvidos de maneira ótima com um conjunto de pontos sobre os quais os itens podem ser posicionados. Vários autores propuseram conjuntos, por exemplo, os padrões normais, *reduced raster points*, padrões normais regulares e padrões meet-in-the-middle. Muito desses conjuntos garantem que uma solução ótima do problema seja encontrada. Investigamos o conjunto proposto em 1987, conhecido como *reduced raster points*, para o qual demonstramos uma prova de otimalidade, demonstramos algumas propriedades, propomos um algoritmo eficiente e apresentamos experimentos que evidenciam sua eficácia sobre outros conjuntos da literatura.

PALAVRAS CHAVE. Problemas de corte e empacotamento. Malhas de pontos. *Reduced raster points*.

PM - Programação Matemática. OC - Otimização Combinatória.

ABSTRACT

Cutting and packing problems involving rectangular-shaped items can be solved optimally with a set of points on which to pack items. Several authors have proposed sets, for example, the normal patterns, the reduced raster points, the regular normal patterns, and the meet-in-the-middle patterns. Many of these sets guarantee an optimal solution to the problem being solved. We investigate the set proposed in 1987, known as reduced raster points, for which we give a formal proof of optimality, demonstrate some properties, propose an efficient algorithm for computing it, and show experiments evidencing their effectiveness over other sets.

KEYWORDS. Cutting and packing problems. Grid of points. *Reduced raster points*.

MP - Mathematical Programming. CO - Combinatorial Optimization.

1. Introduction

In packing problems, items are packed in one (or more bins) to attend an objective to maximize the profit (or minimize the number of bins). In a cutting problem, the bin (or bins) are cut to produce items, observing a given objective. From a theoretical point of view, to cut the bin to make items is the same as packing such items in the bin. It means the terms cut and packing can be used interchangeably, being related to Cutting and Packing problems. Wäscher et al. [2007] proposed a typology for these problems, which considers the kind of objective, the assortment of items and bins, the dimensionality, and the shape of items.

Necessary constraints for cutting and packing problems are the non-overlapping in which items cannot overlap each other, and items must be inside the bin. Additional constraints can also be imposed. They are related to requirements found in real-life situations, as complete shipment, fragility, cargo stability, load balance, among others [Nascimento et al., 2019]. Applications of cutting and packing problems can be found, for example, in production (cutting of bars, steel, glass, wood, leather, and foam) and logistics (loading of pallets, containers, trucks, planes, trains, and ships) [Wäscher et al., 2007; Junqueira and Queiroz, 2019]. Most of these problems are NP-hard [Garey and Johnson, 1979] and well-studied in the operational research area.

For cutting and packing problems in which items and bins have integer rectangular shapes, an optimum solution can be achieved by positioning items on a grid of points representing the bin. The most straightforward set of points is the one with unitary discretization. Herz [1972] proposed a refined set of points, observing that items can be bottom and left-aligned until touching another item or the bin. In other words, this set can be obtained by calculating the conic combinations over the items' dimensions, generating the so-called *canonical dissections*. Later, the canonical dissections were denominated *normal patterns* in an independent work of Christofides and Whitlock [1977]. They proposed a dynamic programming algorithm and showed that this set preserves optimality (that is, there is at least an optimal packing where its items are packed in the considered set).

Terno et al. [1987] introduced the reduced raster points, a set that usually has a much smaller number of points than the set of canonical dissections. In the packing of identical rectangles, Scheithauer and Terno [1996] claimed that the reduced raster points are valid because of dominance considerations. In the general case, Scheithauer [1997] asserted that there is another one having allocation points on the set of reduced raster points for any solution. In Birgin et al. [2010] there is proof of the assertion raised in Scheithauer [1997] for packing identical rectangles. Furthermore, Birgin et al. [2012] asserted, for the general case, that the set of reduced raster points can replace the set of canonical dissections. Queiroz et al. [2015] proved that the set of reduced raster points is sufficient to generate *L*-patterns. These patterns are formed from only *L*-shapes and rectangles. In Almeida Cunha et al. [2019] there is a general proof of the optimality when using the reduced raster points.

The canonical dissections and reduced raster points consider all items of the set of items I . However, Boschetti et al. [2002], following Christofides and Whitlock [1977], reduced the set of normal patterns by observing that each item $i \in I$ can have an individual set of points, which is calculated over $I \setminus \{i\}$. The final set consists of joining each individual set, resulting in the *regular normal patterns*. The set of regular normal patterns' cardinality is never greater than the one of the set of normal patterns. Côté and Iori [2018] proposed how to reduce individual sets, and, consequently, the final set. Such authors presented the set of *meet-in-the-middle* patterns that take a positive integer threshold bounded by the bin's size. This set preserves optimality and dominates (has equal or fewer points than) the regular normal patterns.

The contributions of this work are related to the set of reduced raster points for the general

case of cutting and packing problems: (i) a simple proof that it does preserve optimality; (ii) a presentation of the particular case in which it does not maintain the optimum solution, and an explanation on why this case can be disregarded; (iii) demonstration of some properties that allow reaching an efficient algorithm fashionably for computing it; and, (iv) computational experiments evidencing that this set outperforms others from the literature. The paper is organized as follows: the next section presents the main sets adopted in the literature and how they are defined; Section 3 is devoted to showing that the reduced raster points set preserves optimality, as well as some properties to construct an efficient algorithm; Section 4 brings some computational experiments evidencing that the reduced raster points set overcomes all the other sets, in some cases with a reduction higher than 50%; finally, Section 5 has some concluding remarks.

2. Definitions and Sets of Points

We consider that items and bins have rectangular shapes, for a generic d -dimensional cutting and packing problem. There is a set I with n items, each $i \in I$ has d -dimensions $(l_i, w_i, \dots, h_i)$. There is also a bin that is positioned at $(0, 0, \dots, 0)$, for the Cartesian system, and has d -dimensions $(L, W, \dots, H)$. We assume that all dimensions are positive integer values. If they do not, they can be scaled along with the bin's size, without losing generality, to become integer. Moreover, an item is packed in a d -dimensional point of a grid P , where its bottom-left corner is. A packing of a subset $I' \subseteq I$ must respect the basic constraints of cutting and packing problems, where items have their sides parallel to the bin's sides (orthogonal packing).

In the remaining of the paper, we consider only the first dimension of the items (lengths l_i) and of the bin (length L), such that $0 < l_i \leq L$, for all $i \in I$, and let $l_{\min} = \min\{l_i \mid i \in I\}$. It is important to mention that all the results presented remain valid by considering any other dimension. In what follows, we define the sets of coordinates with canonical dissections [Herz, 1972; Christofides and Whitlock, 1977], reduced raster points [Scheithauer and Terno, 1996], regular normal patterns [Boschetti et al., 2002], and meet-in-the-middle patterns [Côté and Iori, 2018]. As each l_i is assumed to be an integer value, it is true that any integer linear combination of l_i , for $i \in I$, is also integer.

Definition 1. A canonical dissection is a value $c \leq L$ obtained by a non-negative binary linear combination of lengths l_i , for $i \in I$. The set of canonical dissections is:

$$C_L = \{c \in \mathbb{Z} \mid c = \sum_{i \in I} \xi_i l_i, 0 \leq c \leq L, \xi_i \in \{0, 1\}, i \in I\} \quad (1)$$

The dynamic programming recursion in Christofides and Whitlock [1977] can compute the set C_L . More conveniently, we can use the iterative dynamic programming algorithm in Cintra et al. [2008], which worst-case time complexity is $O(nL)$. Herz [1972] proved that the optimal solution is preserved when using the canonical dissections. The terms canonical dissections and normal patterns are used interchangeably in the literature.

Definition 2. A reduced raster point is a canonical dissection $c \in C_L$ for which exists $p \in C_L$ and $c \leq L - p$. The set of reduced raster points is:

$$R_L = \{c \in \mathbb{Z} \mid c = \langle L - p \rangle \text{ for some } p \in C_L\}, \quad (2)$$

where $\langle L - p \rangle = \max\{\bar{c} \in C_L \mid \bar{c} \leq L - p\}$.

A straightforward algorithm to compute the set R_L consists of searching the maximum $c \in C_L$ such that $c \leq L - p$, for each $p \in C_L$. This algorithm has the worst-case time complexity $O(nL + L^2)$, since $O(nL)$ come from computing the set C_L . For the $O(L^2)$, notice that if $l_i = 1$ for all items i , then C_L has L valid coordinates, which in turn requires searching the maximum at most L times.

Remark 1. In the set of reduced raster points, if the sum of all lengths l_i is less than L , the set may be incorrect for generating an optimum solution. Consider the example with $L = 30$ and four items whose lengths are $l_i = (2, 4, 6, 9)$. Then, $C_L = \{0, 2, 4, 6, 8, 9, 10, 11, 12, 13, 15, 17, 19, 21\}$ but $R_L = \{21\}$. Therefore, only the cases in which $L < \sum_{i \in I} l_i$ are interesting. A formal explanation on what causes this incompleteness is given in Section 3.

Definition 3. A regular normal pattern for an item i is a canonical dissection $c \in C_L$ in which i is not part of the conic combination. The set of regular normal patterns for i is:

$$N_{i:L} = \{c \in \mathbb{Z} \mid c = \sum_{j \in I \setminus \{i\}} \xi_j l_j, 0 \leq c \leq L - l_i, \xi_j \in \{0, 1\}, j \in I \setminus \{i\}\} \quad (3)$$

The same algorithm for computing C_L can be used to obtain $N_{i:L}$, if considering the set of items $I \setminus \{i\}$ and the bin's dimension $L - l_i$. The final set of regular normal patterns is $N_L = \cup_{i \in I} N_{i:L}$. A straightforward algorithm to compute N_L has the worst-case time complexity $O(n^2 L)$, since the algorithm of the canonical dissections is called n times.

The idea of computing individual sets of canonical dissections was introduced in Christofides and Whitlock [1977] when they defined the dynamic programming recursion to obtain the overall set of normal patterns. Boschetti et al. [2002] formalized the idea of packing each item only in coordinates from its own set of normal patterns. Remark 2 proves the set of regular normal patterns guarantees an optimal solution.

Remark 2. Let p be a canonical dissection that is generated only if item i is in the conic combination (i.e., $\xi_i = 1$). Notice that i cannot be packed in p because there is no combination that generates p without considering i . Then, i has to be packed in a conical dissection q such that $q < p$. Consequently, the item i does not need to be positioned in the canonical dissections p where $\xi_i = 1$, for all combinations that generate p .

Definition 4. A meet-in-the-middle pattern for an item i , considering a parameter $t \in \{1, 2, \dots, L\}$, is a regular normal pattern $c \in N_L$ if generated from the left to the right due to $c < t$, or it is a “reflected” regular normal pattern if generated from the right to the left due to $c \geq t$. The set of meet-in-the-middle patterns for i and t is defined as the union of the following two sets:

$$M_{i:L,t}^E = \{c \in \mathbb{Z} \mid c = \sum_{j \in I \setminus \{i\}} \xi_j l_j, 0 \leq c \leq \min\{t - 1, L - l_i\}, \xi_j \in \{0, 1\}, j \in I \setminus \{i\}\} \quad (4)$$

$$M_{i:L,t}^D = \{c' \in \mathbb{Z} \mid c' = L - l_i - c, c = \sum_{j \in I \setminus \{i\}} \xi_j l_j, 0 \leq c \leq L - l_i - t, \xi_j \in \{0, 1\}, j \in I \setminus \{i\}\} \quad (5)$$

According to Côté and Iori [2018], the set of meet-in-the-middle patterns for i and t is $M_{i:L,t} = M_{i:L,t}^E \cup M_{i:L,t}^D$. The final set is $M_{L,t} = \cup_{i \in I} M_{i:L,t}$. The parameter t is used to left (for $M_{i:L,t}^E$) or right (for $M_{i:L,t}^D$) align an item depending on where it is packed. We consider a t^* able to give the set $M_L = M_{L,t^*}$ of minimum cardinality when observing all set of individual items, that is $t^* = \operatorname{argmin}_{s \in \{1,2,\dots,L\}} \{ \sum_{i \in I} |M_{i:L,s}| \}$. These authors proved that the set of meet-in-the-middle patterns preserves an optimal solution and presented an $O(n^2L)$ algorithm to compute M_L . They also showed that the meet-in-the-middle patterns and the regular normal patterns are equivalent when $t = L$.

In order to illustrate the cardinality of the sets: canonical dissections C_L , reduced raster points R_L , regular normal patterns N_L , and meet-in-the-middle patterns M_L , it is considered an example with $L = 12$ and four items whose lengths are 2, 4, 6, and 9. Then:

- $C_L = \{0, 2, 4, 6, 8, 9, 10\}$, with 7 coordinates;
- $R_L = \{0, 2, 4, 6, 8, 10\}$, with 6 coordinates;
- $N_L = \{0, 2, 4, 6, 8, 9, 10\}$, with 7 coordinates;
- $M_L = \{0, 2, 3, 4, 6, 8, 10\}$, with 7 coordinates.

We notice that there is no set relationship between the reduced raster points and the meet-in-the-middle patterns (the coordinate 3 is in M_L but not in R_L). The same is true between the reduced raster points and the regular normal patterns. For that, consider an example with $L = 12$ and the items' length 4, 4, and 5, for three items, where $C_L = \{0, 4, 5, 8\}$, $R_L = \{0, 4, 5, 8\}$, and $N_L = \{0, 4, 5\}$, in which 8 is in R_L but not in N_L . Similarly, no set relationship exists between the canonical dissections and the meet-in-the-middle patterns (in the example, 3 is in M_L but not in C_L). On the other hand, the reduced raster points and the regular normal patterns are subsets of the canonical dissections, by definition.

3. Results for the Reduced Raster Points

A new set of points $U_L = \{c \in \mathbb{Z} \mid c = L - \langle L - p \rangle \text{ for some } p \in C_L\}$ is introduced for the proof that R_L preserves an optimum solution. In this new set, the unoccupied length that may occur at the end of the packing is scattered between the items packed. The set U_L preserves an optimum solution according to Proposition 1. Another result is Proposition 2, which links a given set of points G_L (see Definition 5) with its reflected set G'_L .

Proposition 1. *The set U_L preserves an optimal solution.*

Proof. Let $p \in C_L$. Every packing after p , that is, in the sub-bin of size $L - p$, can be arranged such that its length is at most $\langle L - p \rangle$. Notice that every packing from p may generate a residual with the length of at least $(L - p) - \langle L - p \rangle$. Hence, every packing from p is also valid from $p + (L - p - \langle L - p \rangle)$, and then, each $p \in C_L$ can be updated to be $L - \langle L - p \rangle$, resulting in the set U_L . $\square$

Definition 5. Consider an arbitrary set of points G_L that is defined in terms of I . In this set, for each item $i \in I$ and each coordinate $p \in G_L$, if $p + l_i \leq L$, there exists $q \in G_L$ such that $q \geq p + l_i$. Let G'_L be the set with the reflected coordinates under G_L , that is, $G'_L = \{L - p \mid p \in G_L\}$.

Proposition 2. *Every packing in a set G_L can be repacked in its reflected set G'_L .*

Proof. Consider a packing in a given set of points G_L . Suppose that an item i is at $p \in G_L$. Now, consider $q = \min\{r \in G_L \mid r \geq p + l_i\}$, which is the closest coordinate after the right side of i . From definition of G_L , the coordinate q always exists for every item i and coordinate $p \in G_L$. Therefore, consider to repack i in the coordinate $L - q \in G'_L$, repeating this procedure for every item in the packing, resulting in a packing of the same items but in G'_L . $\square$

From Proposition 2, we deduce that for any packing on a set of points G_L , which follows Definition 5, there exists another packing, which is a reflection from the latter, with the same items but rearranged on G'_L . Considering the set U_L and its reflected set U'_L , besides Proposition 2, Theorem 1 shows that there is no loss of generality if using the set R_L .

Theorem 1. *The set of reduced raster points R_L preserves an optimum solution.*

Proof. Let $s = \max\{p \mid p \in C_L\}$. By the definition of C_L and considering Remark 1 in which $L < \sum_{i \in I} l_i$, it is true that $s + l_i > L$ for every item i , otherwise $s + l_i \in C_L$ for some i , and since $s < s + l_i$, then $s < \max\{p \mid p \in C_L\}$, a contradiction. It follows that no item can be packed after the coordinate s , implying on $\langle L - s \rangle = 0$. Then, $L - \langle L - s \rangle = L \in U_L$, implying by Proposition 2 that every packing on U_L can be repacked on U'_L . As U_L preserves an optimum solution (see Proposition 1), then U'_L also preserves an optimum solution. Because $U'_L = \{L - (L - \langle L - p \rangle) \mid p \in C_L\} = \{\langle L - p \rangle \mid p \in C_L\} = R_L$, it follows that R_L preserves an optimum solution. $\square$

In Theorem 1, we consider $L \in U_L$. For this to be true, it is sufficient and necessary that $\max\{p \mid p \in C_L\} + l_i > L$, for each $i \in I$, which depends on $\max\{p \mid p \in C_L\} + l_{\min} > L$. On the other hand, $\max\{p \mid p \in C_L\} + l_{\min} \leq L$ if, and only if, $\sum_{i \in I} l_i + l_{\min} \leq L$. Therefore, the set R_L preserves an optimal solution if we assume $L < \sum_{i \in I} l_i + l_{\min}$, or without loss of generality, $L \leq \sum_{i \in I} l_i$, which explains Remark 1.

In the following, we present some properties of the set of reduced raster points. They are essential to derive a fashionable $O(nL)$ algorithm to compute R_L . In Theorem 2, we prove the set R_L is equivalent to its “reduced” set R_L^R , which is given in Definition 6.

Definition 6. Let $R_L^R = \{\langle L - r \rangle \mid r \in R_L\}$ be the reduced set of the set R_L , that is, this set is generated with only the canonical dissections that are also reduced raster points. For simplicity, the element $\langle L - r \rangle \in R_L^R$ is denoted as $r^{-1} = \langle L - r \rangle$.

Theorem 2. *The set of reduced raster points R_L and the reduced set R_L^R are equivalent.*

Proof. We demonstrate that $R_L^R \subseteq R_L$ and $R_L \subseteq R_L^R$. For $R_L^R \subseteq R_L$, we observe that $R_L \subseteq C_L$ implies on $R_L^R = \{\langle L - r \rangle \mid r \in R_L\} \subseteq \{\langle L - r \rangle \mid r \in C_L\} = R_L$.

For $R_L \subseteq R_L^R$, let $r = \langle L - p \rangle \in R_L$, for some $p \in C_L$. We show that exists $q \in R_L$, such that $r = \langle L - q \rangle$, that is for all $p \in C_L$ there exists $q \in R_L$ such that $\langle L - p \rangle = \langle L - q \rangle$. It is sufficient to find $q \in R_L$ such that $p \leq q \leq L - r$, since $\langle L - p \rangle = r = \langle L - (L - r) \rangle$. Notice that $\langle L - r \rangle$ is such that $p \leq \langle L - r \rangle \leq L - r$. Then, if we choose $q = \langle L - r \rangle$, then $r = \langle L - p \rangle = \langle L - \langle L - r \rangle \rangle$. $\square$

Theorem 2 shows that the set of reduced raster points is sufficient to generate itself. It may seem recursive, but it leads to the following results, including an efficient algorithm to compute the set R_L . An immediate consequence of this theorem is Corollary 1.

Corollary 1. *For all reduced raster points coordinates $r \in R_L$, $r = (r^{-1})^{-1}$.*

Proof. Notice from Theorem 2 that $r = \langle L - \langle L - r \rangle \rangle = \langle L - r^{-1} \rangle = (r^{-1})^{-1}$. □

We observed a one-to-one correspondence between two coordinates of reduced raster points, r and r^{-1} , if they are in the first half of L . This result is given in Theorem which takes into consideration Proposition 3.

Proposition 3. *If a reduced raster points coordinate $r \in R_L$ satisfies $r > \frac{L}{2}$, then $r^{-1} \leq \frac{L}{2}$.*

Proof. Notice that $L - r < \frac{L}{2}$ implies on $r^{-1} = \langle L - r \rangle < \frac{L}{2}$. □

Theorem 3. *Let $r \in R_L$ be a reduced raster points coordinate. Then, $r = r^{-1}$ if, and only if, $r \leq \frac{L}{2}$ and $r^{-1} \leq \frac{L}{2}$.*

Proof. First, we suppose $r = r^{-1}$, for $r > \frac{L}{2}$. From Proposition 3, $r^{-1} \leq \frac{L}{2}$, implying on $r \neq r^{-1}$, a contradiction. Then, $r = r^{-1}$ implies on $r \leq \frac{L}{2}$ and $r^{-1} \leq \frac{L}{2}$.

Now, we suppose $r \leq \frac{L}{2}$ and $r^{-1} \leq \frac{L}{2}$. As $L - r \geq \frac{L}{2}$, then $r^{-1} = \langle L - r \rangle \geq \langle \frac{L}{2} \rangle$. Because $r^{-1} \leq \frac{L}{2}$, so $r^{-1} \leq \langle \frac{L}{2} \rangle$ and, then, $r^{-1} = \langle \frac{L}{2} \rangle$. Analogously, we show that $r = \langle \frac{L}{2} \rangle$, proving that $r = \langle \frac{L}{2} \rangle = r^{-1}$. Therefore, from $r \leq \frac{L}{2}$ and $r^{-1} \leq \frac{L}{2}$, we have $r = r^{-1}$. □

Corollary 2. *If a reduced raster points coordinate $r \in R_L$ satisfies $r = r^{-1}$, then r is unique.*

Proof. Notice from Theorem 3 if $r = r^{-1}$, then $r = r^{-1} = \langle \frac{L}{2} \rangle$. □

From Theorem 3, for each $r \in R_L$ such that $r \leq \frac{L}{2}$, we have either $r^{-1} = r$ or $r^{-1} > \frac{L}{2}$. From Proposition 3, for each $r \in R_L$ such that $r > \frac{L}{2}$, we have $r^{-1} \leq \frac{L}{2}$. Then, we can redefine the set of reduced raster points as $R_L = \{r^{-1} \mid r > \frac{L}{2}, r \in R_L\} \cup \{r^{-1} \mid r \leq \frac{L}{2}, r \in R_L\}$. Considering Corollary 1 and Theorem 3, it follows that $\{r^{-1} \mid r > \frac{L}{2}, r \in R_L\} \subseteq \{(r^{-1})^{-1} \mid r \leq \frac{L}{2}, r \in R_L\}$, motivating Remark 3.

Remark 3. The set of reduced raster points can be defined as $R_L = \{(r^{-1})^{-1} \mid r \leq \frac{L}{2}, r \in R_L\} \cup \{r^{-1} \mid r \leq \frac{L}{2}, r \in R_L\}$. Moreover, as $R_L \subseteq C_L$, then $R_L = \{(r^{-1})^{-1} \mid r \leq \frac{L}{2}, r \in C_L\} \cup \{r^{-1} \mid r \leq \frac{L}{2}, r \in C_L\}$.

Remark 3 shows that is not necessary to consider all canonical dissections in C_L for generating the set of reduced raster points. It is necessary to consider only coordinates $p \in C_L$ such that $p \leq \frac{L}{2}$, for generating $r^{-1} = \langle L - p \rangle$ and $(r^{-1})^{-1} = \langle L - \langle L - p \rangle \rangle$. Moreover, if the canonical dissections p are sorted in non-decreasing order, then from Corollary 2, it is sufficient to consider only $p \leq \langle \frac{L}{2} \rangle$. A further improvement is achieved by observing that, for each $p \in C_L$, we calculate both $\langle L - p \rangle$ and $\langle L - \langle L - p \rangle \rangle$. However, for each $q \in [p, \langle L - \langle L - p \rangle \rangle]$, we have $\langle L - q \rangle = \langle L - p \rangle$. Therefore, it is not necessary to calculate $\langle L - q \rangle$ and $\langle L - \langle L - q \rangle \rangle$, neither to consider canonical dissections $q \in C_L$ such that $q > \langle L - \langle L - p \rangle \rangle$.

To the best of our knowledge, no algorithm for calculating the set of reduced raster points has been described in the literature. As commented in Section 2, the trivial algorithm is $O(nL + L^2)$. From Remark 3 and observing that $\max\{p \in C_L \mid p \leq L - c\}$ has a dynamic programming structure, since, for each $x = 0, \dots, L$, the value of $\max\{p \in C_L \mid p \leq x\}$ can be stored at the x -th position of a vector if $x \in C_L$, otherwise, the x -th position assumes the same value in the $(x - 1)$ -th position, we introduce Algorithm 1 to compute the set of reduced raster points. This algorithm has the worst-case time complexity $O(L)$ when canonical dissections are known.

Algorithm 1: Obtaining the set of reduced raster points.

Input: set of canonical dissections C_L ; length L of the bin.
Output: set of reduced raster points R_L .

```

1  $R_L \leftarrow$  an empty set.
2  $K \leftarrow$  an empty vector of size  $L+1$ .
3 for  $x \leftarrow 0$  to  $L$  do
4   if  $x \in C_L$  then
5      $K[x] \leftarrow x$ .
6   else
7      $K[x] \leftarrow K[x-1]$ .
8  $last \leftarrow 0$ .
9 for  $x \leftarrow 0$  to  $K[\lceil L/2 \rceil]$  do
10  if  $x \in C_L$  and  $x > last$  then
11     $R_L \leftarrow R_L \cup \{K[L-x], K[L-K[L-x]]\}$ .
12     $last \leftarrow K[L-K[L-x]]$ .
```

4. Numerical Tests

Computational experiments are reported to show the cardinality of the sets defined in Section 2. All algorithms were coded in the C++ language. The experiments were conducted on a computer with Linux Ubuntu 14.04 LTS as the operating system, Intel Core i5 with 3.4 GHz and 8 GB of RAM.

For the experiments, we consider all instances used in Birgin et al. [2012], for a two-dimensional cutting and packing problem. Then, the sets are defined as $C_{L,W} = \{(p, q) \mid p \in C_L, q \in C_W\}$, $R_{L,W} = \{(p, q) \mid p \in R_L, q \in R_W\}$, $N_{L,W} = \{(p, q) \mid p \in N_L, q \in N_W\}$, and $M_{L,W} = \{(p, q) \mid p \in M_L, q \in M_W\}$. The instances under consideration are GCUT1-GCUT17, H, HZ1, HZ2, M1-M5, MW1-MW5, U1-U4, W1-W4, LU1-LU4, LW1-LW4, UU1-UU11, UW1-UW11, APT10-APT20, totaling 95 instances.

The comparison is made over the final sets, disregarding the analysis of individual sets for each item. For each instance and each set, we calculate the reduction (in percentage) of the cardinality of a given set comparing with the set of canonical dissections $C_{L,W}$, which is $\frac{|C_{L,W}| - |Set_{L,W}|}{|C_{L,W}|} \times 100\%$. The reductions are in Table 1, while detailed information is in Table 2. Table 1 contains:

- *Best*: the percentage of instances with a reduction that is at least as large as the largest reduction among all the sets;
- *Mean*: the mean of the reduction, for the set;
- *Std. Dev.*: the standard deviation of the reduction, for the set;
- *Max*: the largest reduction, for the set;
- *Min*: the smallest reduction, for the set.

Observing the “Best” row in Table 1, no set has cardinality smaller than the one of $R_{L,W}$, for all the instances. The set $M_{L,W}$ has the same cardinality of $R_{L,W}$ for 40% of the instances.

Table 1: Results of the different sets compared with the set of canonical dissections.

Grid	$N_{L,W}$	$M_{L,W}$	$R_{L,W}$
Best	0.00%	40.00%	100.00%
Mean	2.68%	57.11%	61.31%
Std. Dev.	5.83%	28.82%	29.88%
Max	25.55%	94.90%	96.83%
Min	0.00%	0.88%	0.88%

The set $N_{L,W}$ always has cardinality greater than those of $M_{L,W}$ and $R_{L,W}$. Moreover, the set $R_{L,W}$ has the smallest cardinality for 60% of the instances. For the other 40% instances, $R_{L,W}$ has cardinality at least as small as all others.

In the “Mean” row of Table 1, we observe the mean percentage reduction for the sets $R_{L,W}$ and $M_{L,W}$ are relatively high, both with an average which is less than half of the set $C_{L,W}$. On the other hand, the mean reduction for $N_{L,W}$ is not high, being only 2.68%. In the “Std. Dev.” row, the standard deviation for the reduction of $N_{L,W}$ is not very high, which means that most of the reductions for this set are close to the mean. The standard deviation for $M_{L,W}$ and $R_{L,W}$, which are very similar, are significantly large, showing the reductions for these sets may vary more. For the “Max” row, the maximum reduction for $N_{L,W}$ is 25.55%, while it is 94.90%, for $M_{L,W}$, and 96.83%, for $R_{L,W}$. When analyzing the “Min” row, no reduction on the cardinality happens for $N_{L,W}$, while for $M_{L,W}$ and $R_{L,W}$, there is always a reduction for every instance.

For the instances analyzed, we conclude that the set of reduced raster points is the best overall. About problems and computational results using the reduced raster points, we can refer to Birgin et al. [2012] and Queiroz et al. [2012] for, respectively, two- and three-dimensional cutting and packing problems. It is worth mentioning that Côté and Iori [2018] solved one- and two-dimensional packing problems with the meet-in-the-middle patterns, under which further extensions by using the reduced raster points can be made.

5. Concluding Remarks

A simple proof that packing in the reduced raster points preserves an optimum solution is given. Besides, properties involving such points are presented, allowing us to achieve an efficient algorithm for them. This algorithm has a better worst-case time complexity, and the set of reduced raster points generally has a smaller cardinality compared to the others. We also show no set relationship between the set of reduced raster points and the sets of regular normal patterns and meet-in-the-middle patterns.

The set of reduced raster points may be the best choice for cutting and packing problems and even for other combinatorial optimization problems (as scheduling problems). This motivates future research involving such points, including developing reduction criteria to improve them [Boschetti et al., 2002; Côté and Iori, 2018].

Acknowledgements

The authors acknowledge the National Council for Scientific and Technological Development (CNPq - grant 308312/2016-3) and the State of Goiás Research Foundation (FAPEG) for their financial support.

Table 2: Cardinality of the set of canonical dissections and reduction of the other sets over it for each instance.

Instance	$C_{L,W}$	$N_{L,W}$	$M_{L,W}$	$R_{L,W}$
apt10	2225880	0.00%	26.26%	26.26%
apt11	2662464	0.00%	24.24%	24.24%
apt12	2178010	0.00%	61.85%	61.96%
apt13	2169132	0.00%	24.46%	24.46%
apt14	2358648	0.00%	38.25%	38.25%
apt15	3242123	0.00%	37.50%	37.50%
apt16	3920490	0.00%	39.30%	39.30%
apt17	2780694	0.00%	26.72%	26.72%
apt18	2472082	0.00%	61.59%	61.69%
apt19	3118412	0.00%	53.93%	53.93%
apt20	4390615	0.00%	36.38%	36.38%
apt21	2505771	0.00%	40.63%	40.63%
apt22	3201985	0.00%	33.48%	33.48%
apt23	2558772	0.00%	41.54%	41.54%
apt24	3302250	0.00%	29.60%	29.60%
apt25	2444172	0.00%	47.01%	47.01%
apt26	2075606	0.00%	41.11%	41.11%
apt27	1772340	0.00%	40.97%	40.97%
apt28	3394965	0.00%	26.84%	26.84%
apt29	2960048	0.00%	33.70%	33.70%
b1	3570075	0.00%	49.98%	50.06%
b2	3696678	0.00%	43.72%	43.72%
b3	3623724	0.00%	46.87%	46.87%
b4	3747310	0.00%	43.94%	43.94%
b5	3505696	0.00%	49.84%	49.92%
b6	3679731	0.00%	41.51%	41.51%
b7	5056656	0.00%	13.31%	13.31%
gcut1	176	0.00%	68.18%	72.73%
gcut2	1598	2.94%	70.28%	79.85%
gcut3	3080	0.00%	65.10%	65.10%
gcut4	6720	0.00%	66.52%	68.57%
gcut5	330	0.00%	56.36%	72.73%
gcut6	1184	6.25%	71.62%	84.21%
gcut7	1984	15.63%	64.06%	80.04%
gcut8	12408	0.00%	79.43%	79.90%
gcut9	250	0.00%	36.00%	66.40%
gcut10	1296	25.46%	75.69%	85.57%
gcut11	6386	0.00%	84.12%	90.68%
gcut12	16758	0.68%	87.52%	88.75%
gcut13	3086622	0.00%	68.19%	70.43%
gcut14	6654660	0.00%	35.81%	35.81%
gcut15	6985114	0.00%	32.89%	32.89%
gcut16	7461675	0.00%	26.28%	26.28%
gcut17	7832979	0.00%	22.21%	22.21%
h	420	9.52%	14.29%	62.86%
hz1	658	17.93%	21.58%	53.50%
hz2	432	3.70%	13.19%	37.50%
lu1	223162368	0.00%	7.53%	7.53%

Instance	$C_{L,W}$	$N_{L,W}$	$M_{L,W}$	$R_{L,W}$
lu2	806230236	0.00%	7.54%	7.54%
lu3	502435500	0.00%	1.63%	1.63%
lu4	401143050	0.00%	0.88%	0.88%
lw1	223162368	0.00%	7.53%	7.53%
lw2	806230236	0.00%	7.54%	7.54%
lw3	502435500	0.00%	1.63%	1.63%
lw4	401143050	0.00%	0.88%	0.88%
m1	918	5.88%	66.67%	75.60%
m2	2739	7.19%	72.07%	79.04%
m3	1495	25.55%	74.72%	83.28%
m4	1518	11.46%	76.48%	88.93%
m5	1978	15.07%	78.77%	89.48%
mw1	918	5.88%	66.67%	75.60%
mw2	2739	7.19%	72.07%	79.04%
mw3	1495	25.55%	74.72%	83.28%
mw4	1932	15.22%	78.26%	90.06%
mw5	1978	15.07%	78.77%	89.48%
u1	373620	6.76%	78.92%	85.20%
u2	43602	3.48%	80.00%	91.06%
u3	6757408	0.12%	94.24%	96.16%
u4	25142046	0.00%	47.41%	47.41%
uu1	6365	0.00%	79.89%	85.29%
uu2	23572	3.40%	89.67%	91.98%
uu3	17270	1.54%	88.65%	91.93%
uu4	62424	0.65%	90.04%	92.94%
uu5	139093	0.00%	89.29%	91.70%
uu6	43560	1.38%	92.01%	93.93%
uu7	216140	0.00%	88.72%	92.12%
uu8	168989	0.21%	93.45%	94.78%
uu9	238170	0.21%	92.83%	94.31%
uu10	458052	0.35%	93.66%	96.10%
uu11	3727794	0.38%	86.04%	88.48%
uw1	9270	3.17%	83.17%	90.33%
uw2	35717	0.00%	84.71%	87.87%
uw3	24708	1.41%	85.14%	89.95%
uw4	109900	0.00%	86.57%	88.50%
uw5	34400	2.15%	88.87%	92.47%
uw6	140980	0.00%	92.69%	94.45%
uw7	179894	0.00%	92.72%	94.72%
uw8	366183	0.36%	93.00%	95.08%
uw9	179607	1.44%	94.65%	96.83%
uw10	589014	0.00%	92.96%	96.48%
uw11	85173	0.00%	60.39%	61.82%
w1	4970884	0.00%	85.70%	88.41%
w2	288036	10.96%	94.90%	96.58%
w3	12969242	0.00%	85.94%	87.16%
w4	32849336	0.00%	30.68%	30.68%

References

- Almeida Cunha, J. G., de Lima, V. L., and Queiroz, T. A. (2019). Grids for cutting and packing problems: a study in the 2d knapsack problem. *4OR - A Quarterly Journal of Operations Research*, in press:1–47.
- Birgin, E. G., Lobato, R. D., and Morabito, R. (2010). An effective recursive partitioning approach for the packing of identical rectangles in a rectangle. *Journal of the Operational Research Society*, 61(2):306–320.
- Birgin, E. G., Lobato, R. D., and Morabito, R. (2012). Generating unconstrained two-dimensional non-guillotine cutting patterns by a recursive partitioning algorithm. *Journal of the Operational Research Society*, 63(2):183–200.
- Boschetti, M. A., Mingozzi, A., and Hadjiconstantinou, E. (2002). New upper bounds for the two-dimensional orthogonal non-guillotine cutting stock problem. *IMA Journal of Management Mathematics*, 13(2):95.
- Christofides, N. and Whitlock, C. (1977). An algorithm for two dimensional cutting problems. *Operations Research*, 25:30–44.
- Cintra, G. F., Miyazawa, F. K., Wakabayashi, Y., and Xavier, E. C. (2008). Algorithms for two-dimensional cutting stock and strip packing problems using dynamic programming and column generation. *European Journal of Operational Research*, 191:59–83.
- Côté, J.-F. and Iori, M. (2018). The meet-in-the-middle principle for cutting and packing problems. *INFORMS Journal on Computing*, 30(4):625–786.
- Garey, M. R. and Johnson, D. S. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman, San Francisco.
- Herz, J. C. (1972). Recursive computational procedure for two-dimensional stock cutting. *IBM Journal of Research and Development*, 16(5):462–469.
- Junqueira, L. and Queiroz, T. A. (2019). The static stability of support factor-based rectangular packings: an assessment by regression analysis. *International Transactions in Operational Research*, in press:1–23.
- Nascimento, O. X., Queiroz, T., and Junqueira, L. (2019). A MIP-CP based approach for two- and three-dimensional cutting problems with staged guillotine cuts. *Annals of Operations Research*, in press:1–31.
- Queiroz, T. A., Miyazawa, F. K., and Wakabayashi, Y. (2015). On the l -approach for generating unconstrained two-dimensional non-guillotine cutting patterns. *4OR*, 13(2):199–219.
- Queiroz, T. A., Miyazawa, F. K., Wakabayashi, Y., and Xavier, E. C. (2012). Algorithms for 3D guillotine cutting problems: unbounded knapsack, cutting stock and strip packing. *Computers & Operations Research*, 39:200–212.
- Scheithauer, G. (1997). Equivalence and dominance for problems of optimal packing of rectangles. *Ricerca Operativa*, 27:3–34.

- Scheithauer, G. and Terno, J. (1996). The G4-heuristic for the pallet loading problem. *Journal of the Operational Research Society*, 47:511–522.
- Terno, J., Lindemann, R., and Scheithauer, G. (1987). Zuschnittprobleme and ihre praktische lösung. Technical Report, Verlag Harry Deutsch, Thun und FrankfurtMain.
- Wäscher, G., Haussner, H., and Schumann, H. (2007). An improved typology of cutting and packing problems. *European Journal of Operational Research*, 183(3):1109–1130.