

Identifying Subclasses of Helly B_1 -EPG Graphs

Liliana Alcón

National University of La Plata - UNLP
Department of Mathematics, College of Exact Sciences, Street 50 and 115,
La Plata, Buenos Aires, Argentina, B1900
`liliana@mate.unlp.edu.ar`

María Pía Mazzoleni

National University of La Plata - UNLP
Department of Mathematics, College of Exact Sciences, Street 50 and 115,
La Plata, Buenos Aires, Argentina, B1900
`pia@mate.unlp.edu.ar`

Tanilson Dias dos Santos

Federal University of Rio de Janeiro, PESC/COPPE/UFRJ
Centro de Tecnologia - Av. Horácio Macedo, 2030 - 101 - Cidade Universitária,
Rio de Janeiro - RJ, 21941-450
Federal University of Tocantins - UFT
Quadra 109 Norte, Avenida NS-15, ALCNO-14, Plano Diretor Norte,
Palmas-TO, 77001-090
`tanilson.dias@uft.edu.br`

RESUMO

Este trabalho apresenta estruturas presentes em qualquer subgrafo induzido B_1 -EPG que não admita uma representação B_1 -EPG-Helly. Por meio dessas estruturas foi possível delimitar alguns conjuntos de subgrafos que definem subfamílias Helly. Em particular, essa pesquisa apresenta uma caracterização para algumas famílias não triviais de grafos que estão propriamente contidas na classe de grafos B_1 -EPG-Helly. Nomeadamente essas famílias são as dos grafos Bipartidos, Blocos, Cactus e Linha de Bipartido.

PALAVRAS CHAVE. EPG, Propriedade Helly, Dobra Simples.

TAG - Teoria e Algoritmos em Grafos

ABSTRACT

This research presents structures that are always present in B_1 -EPG graphs that do not admit a Helly B_1 -EPG representation and thus we give some sets of subgraphs that delimit Helly subfamilies. In particular, this paper presents characterizations for some non-trivial families of graphs properly contained in Helly B_1 -EPG, namely Bipartite, Block, Cactus and Line of Bipartite graphs.

KEYWORDS. EPG, Helly property, Single Bend.

TGA - Theory and Graph Algorithms

1. Introduction

Edge intersection graphs of paths on a grid are called *EPG graphs*.

In [Golumbic et al., 2009], the authors proved that every graph is EPG, and started the study of the subclasses defined by bounding the number of times any path used in the representation can bend. Graphs admitting a representation where paths have at most k changes of direction (bends) were called B_k -EPG graphs. In particular, when the paths have at most one bend we have the B_1 -EPG graphs or a *single bend EPG graphs*.

A collection of sets satisfies the *Helly property* when every pairwise intersecting sub-collection has at least one common element. When this property is satisfied by the set of vertices (edges) of the paths used in a representation, we get a Helly representation. Helly B_1 -EPG graphs were studied in [Bornstein et al., 2019]. It is known that not every B_1 -EPG graph admits a Helly B_1 -EPG representation. We are interested in determining the subgraphs that make B_1 -EPG graphs do not admit a Helly representation. In the present work, we describe some structures that will be present in any such subgraph, and, in addition, we present new Helly B_1 -EPG subclasses. Moreover, we describe new Helly B_1 -EPG subclasses and we give some sets of subgraphs that delimit Helly subfamilies.

In the present paper we will explore B_1 -EPG graphs, in particular diamond-free graphs and Chordal graphs. We will work on the question about the containment relation between Block, Cactus, Line of Bipartite and Helly B_1 -EPG graph classes.

2. Definitions and Technical Results

The *vertex set* and the *edge set* of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Given a vertex $v \in V(G)$, $N(v)$ and $N[v]$ represent the open and the close *neighborhood* of v in G , respectively. For a subset $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S . If $\mathcal{F}$ is any family of graphs, we say that G is $\mathcal{F}$ -free if G has no induced subgraph isomorphic to a member of $\mathcal{F}$. A *cycle*, denoted by C_n , is a sequence of distinct vertices $v_1, \dots, v_n, v_1$ where $v_i \neq v_j$ for $i \neq j$ and $(v_i, v_{i+1}) \in E(G)$, such that $n \geq 3$. A *chord* is an edge that is between two non-consecutive vertices in a sequence of vertices of a cycle. An *induced cycle* or *chordless cycles* is a cycle that has no chord, in this paper an induced cycle will simply be called *cycle*. A graph G formed by an induced cycle H plus a single universal vertex v connected to all vertices of H is called *wheel graph*. If the wheel has n vertices, it is denoted by n -wheel.

A graph is a B_k -EPG graph if it admits an EPG representation in which each path has at most k bends. When $k = 1$ we say that this is a *single bend EPG* representation or simply a B_1 -EPG representation. A *clique* is a set of pairwise adjacent vertices and an *independent set* is a set of pairwise non adjacent vertices. Given an EPG representation of a graph G , we will identify each vertex v of G with the corresponding path P_v of the grid used in the representation. Accordingly, for instance, we will say that a vertex of G covers or contains some edge of the grid (meaning that the corresponding path does), or that a set of paths of the representation induces a subgraph of G (meaning that the corresponding set of vertices does).

In a B_1 -EPG representation, a clique K is said to be an *edge-clique* if all the vertices of K share a common edge of the grid (see Figure 1(a)). A *claw of the grid* is a set of three edges of the grid incident into a same point of the grid, which is called the *center of the claw*. The two edges of the claw that have the same direction form the *base of the claw*. If K is not an edge-clique, then there exists a claw of the grid (and only one) such that the vertices of K are those containing exactly two of the three edges of the claw; such a clique is called *claw-clique* [Golumbic et al., 2009] (see Figure 1(b)).

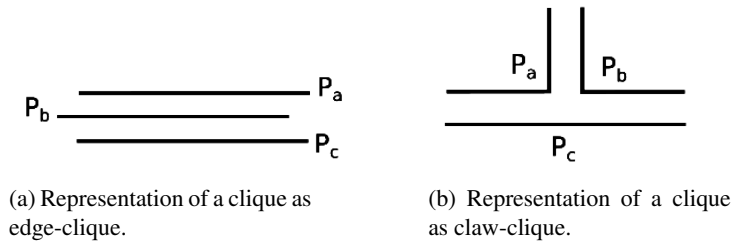
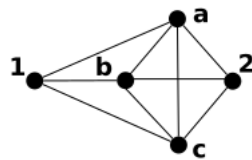


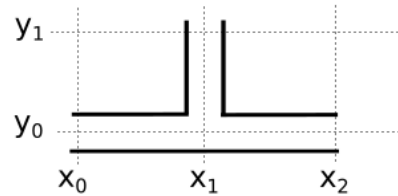
Figure 1: Examples of clique representations.

Notice that if three vertices induce a claw-clique, then exactly two of them turn at the center of the corresponding claw of the grid, and the third one contains the base of the claw. Furthermore, any other vertex adjacent to the three must contain two of the edges of that claw, then the following lemma holds.

Lemma 1. *If three vertices are together in more than one maximal clique of a graph G , then in any B_1 -EPG representation of G the three vertices do not form a claw-clique.*



(a) Example of two maximal cliques sharing vertices.



(b) Representation of a claw-clique in grid.

Figure 2: Vertices represented by a claw are present in a unique maximal clique.

In [Ries, 2009] Asinowski et al. proved the following lemma for C_4 -free graphs.

Lemma 2. [Ries, 2009] *Let G be a B_1 -EPG graph. If G is C_4 -free, then there exists a B_1 -EPG representation of G such that every maximal claw-clique K is represented on a claw of the grid whose base is covered only by vertices of K .*

We have obtained the following similar result for diamond-free graphs. A *diamond* is a graph G with vertex set $V(G) = \{a, b, c, d\}$ and edge set $E(G) = \{ab, ac, bc, bd, cd\}$ (see Figure 3).

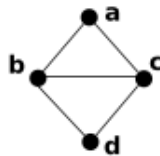


Figure 3: Diamond graph.

Lemma 3. *Let G be a B_1 -EPG graph. If G is diamond-free, then in any B_1 -EPG representation of G , every maximal claw-clique K is represented on a claw of the grid whose edges are covered only by vertices of K .*

Proof. Let K be a maximal clique which is a claw-clique in a given B_1 -EPG representation of G . Then there exist three vertices of K which induce a claw-clique K' on the same claw of the grid than K . Assume, in order to derive a contradiction, that a vertex $v \notin K$ covers some edges of the claw. Clearly, v must cover only one of such edges. Therefore v and the vertices of K' induce a diamond, a contradiction. $\square$

Let Q be a grid and let $(a_1, b), (a_2, b), (a_3, b), (a_4, b)$ be a 4-star centered at b as depicted in Figure 4(a). Let $\mathcal{P} = \{P_1, \dots, P_4\}$ be a collection of four paths each containing a different pair of edges of the 4-star. Following [Golumbic et al., 2009], we say that the four paths form

- a *true pie* when each one has a bend at b , Figure 4(b); and
- a *false pie* when exactly two of the paths bend at b and they do not share an edge of the 4-star, Figure 4(c).

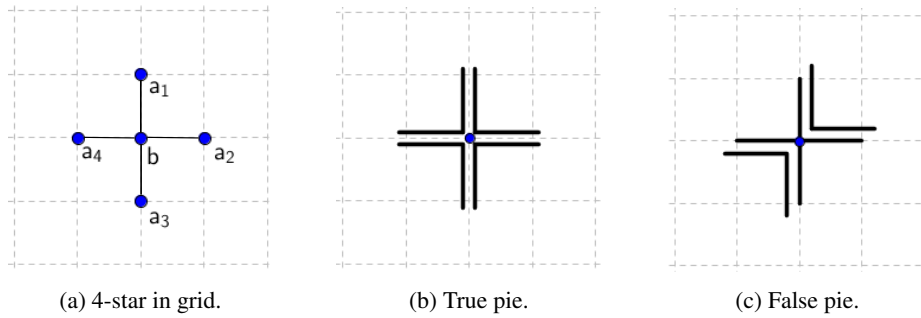


Figure 4: B_1 -EPG representation of the induced cycle of size 4 as pies with emphasis in center b .

Clearly if four paths of a B_1 -EPG representation of G form a pie, then the corresponding vertices induce a 4-cycle in G . The following result can be easily proved. We say that a set of paths form a claw when each pair of edges of the claw is covered by some of the paths.

Lemma 4. *In any B_1 -EPG representation of a graph G , a set of paths forming two different claws centered at the same point of the grid contains four paths forming either a true pie or a false pie. Therefore, in any B_1 -EPG representation of a chordal graph G , no two maximal claw-cliques of G are centered at the same point of the grid.*

Lemma 5. *Let G be a graph whose vertex set can be partitioned into a non trivial clique K and an independent set $I = \{w_1, w_2, w_3\}$, such that each vertex of K is adjacent to each vertex of I . Then, in any B_1 -EPG representation of G , at least one of the cliques $K_i = K \cup \{w_i\}$, with $1 \leq i \leq 3$, is an edge-clique.*

Proof. Assume, in order to derive a contradiction, that the three cliques are claw-cliques. By Lemma 4, they have different centers, say the points q_1, q_2, q_3 of the grid, respectively. Since at least two paths have a bend at the center of a claw, for each $i \in \{1, 2, 3\}$, there must exist a vertex v_i of K such that the corresponding path P_{v_i} turns at the point q_i of the grid. Notice that each one of the three paths P_{v_i} must contain the three grid points q_1, q_2 and q_3 . To prove that this is not possible, we will consider, without loss of generality, two cases. First, q_1 is between q_2 and q_3 in P_{v_1} . Then, P_{v_3} cannot turn at q_3 and contain q_1 and q_2 . And second, q_2 is between q_1 and q_3 in P_{v_1} . In this case, P_{v_2} cannot turn at q_2 and contain q_1 and q_3 ; thus the proof is completed. $\square$

3. Subclasses of Helly B_1 -EPG Graphs

In this section, we delimit some subclasses of B_1 -EPG graphs that admit a Helly B_1 -EPG representation. It is known that B_1 -EPG and Helly B_1 -EPG are hereditary classes, so they can be characterized by forbidden structures. In both cases, finding the list of minimal forbidden induced subgraphs are challenging open problems. Taking a step towards solving those problems, we describe a few structures at least one of which will necessarily be present in any B_1 -EPG graph that does not admit a Helly representation. In addition, we show that the well known families of Block graphs, Cactus and Line of Bipartite graphs are totally contained in the class Helly B_1 -EPG.

Let $S_3, S_{3'}, S_{3''}$ and C_4 be the graphs depicted in Figure 6.

Theorem 6. *Let G be a B_1 -EPG graph. If G is $\{S_3, S_{3'}, S_{3''}, C_4\}$ -free then G is a Helly B_1 -EPG graph.*

Proof. If G is not a Helly B_1 -EPG graph, then in each B_1 -EPG representation of G , there is at least one clique that is represented as claw-clique and no as edge-clique. Consider any B_1 -EPG representation of G and let K be a maximal clique which is represented as a claw-clique. Assume, w.l.o.g, K is on a claw of the grid with base $[x_0, x_2] \times \{y_0\}$ and center $C = (x_1, y_0)$. Denote by $\mathcal{P}_K$ the set of paths corresponding to the vertices of K . By Lemma 2, the grid segment $[x_0, x_2] \times \{y_0\}$ is covered only by vertices of K .

For every $\lrcorner$ -path (resp. $\llcorner$ -path) belonging to $\mathcal{P}_K$, we do the following: if the path does not intersect any path $P_t \notin \mathcal{P}_K$ on column x_1 , then we delete its vertical segment and add the grid segment $[x_1, x_2] \times \{y_0\}$ (resp. $[x_0, x_1] \times \{y_0\}$). If after this transformation there is no more $\lrcorner$ -paths (resp. $\llcorner$ -paths) in $\mathcal{P}_K$, then we are done since we have obtained an edge-clique. So we may assume that every $\lrcorner$ -path and every $\llcorner$ -path in $\mathcal{P}_K$ intersects some path $P_t \notin \mathcal{P}_K$ on column x_1 (notice that we can assume is the same path P_t for all the vertices).

Now, if none of the $\lrcorner$ -paths belonging to $\mathcal{P}_K$ intersects a path non in $\mathcal{P}_K$ on the line y_0 , then we can replace the horizontal part of those paths by the segment $[x_1, x_2] \times \{y_0\}$, getting an edge representation of the clique K . Thus, we can assume there exists at least one $\lrcorner$ -path $P_v \in \mathcal{P}_K$ intersecting some path $P_{v'} \notin \mathcal{P}_K$ on line y_0 . Analogously, there exists at least one $\llcorner$ -path $P_{v''} \in \mathcal{P}_K$ intersecting some path $P_{t''} \notin \mathcal{P}_K$ on line y_0 , as depicted in Figure 5. Notice that vertex t' cannot be adjacent to any of the vertices t, v' or t'' ; and, in addition, vertex t'' cannot be adjacent to t , or v .

Finally, since K is claw-clique, there is a path $P_u \in \mathcal{P}_K$ covering the base of the claw. Depending on the possibles adjacencies between u and t' or t'' , one of the graphs $S_3, S_{3'}$ or $S_{3''}$ is obtained.

□

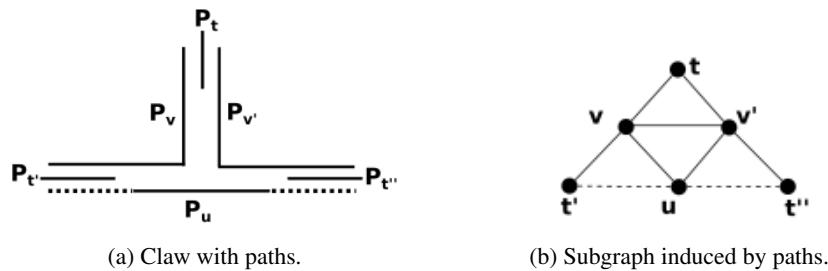


Figure 5: Reconstruction of the intersection model.

Notice that any bull-free graph is $\{S_3, S_{3'}, S_{3''}\}$ -free, so our previous result implies Lemma 5 of [Ries, 2009].

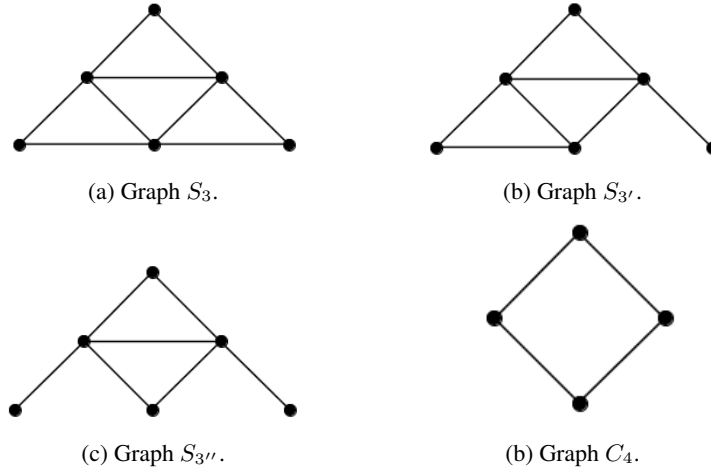


Figure 6: Graphs on the statement of Theorem 6.

Next theorem has as consequence the identification of several graph classes where the existence of a B_1 -EPG representation ensures the existence of a Helly B_1 -EPG representation.

Theorem 7. *If G is a B_1 -EPG and diamond-free graph then G is a Helly B_1 -EPG graph.*

Proof. If G is not a Helly B_1 -EPG graph, then in each B_1 -EPG representation of G , there is at least one clique that is represented as claw-clique and no as edge-clique. Consider any B_1 -EPG representation of G and let K be a maximal clique which is represented as a claw-clique. Assume, w.l.o.g, K is on a claw of the grid with base $[x_0, x_2] \times \{y_0\}$ and center $C = (x_1, y_0)$. Denote by $\mathcal{P}_K$ the set of paths corresponding to the vertices of K . By Lemma 3, the grid segment $[x_0, x_2] \times \{y_0\}$ is covered only by vertices of K . For every $\lrcorner$ -path (resp. $\llcorner$ -path) belonging to $\mathcal{P}_K$, we do the following: if the path does not intersect any path $P_t \notin \mathcal{P}_K$ on column x_1 , then we delete its vertical segment and add the grid segment $[x_1, x_2] \times \{y_0\}$ (resp. $[x_0, x_1] \times \{y_0\}$). If after this transformation there is no more $\lrcorner$ -paths (resp. $\llcorner$ -paths) in $\mathcal{P}_K$, then we are done since we have obtained an edge-clique. So we may assume that every $\lrcorner$ -path and every $\llcorner$ -path in $\mathcal{P}_K$ intersects some path $P_t \notin \mathcal{P}_K$ on column x_1 (notice that we can assume is the same path P_t for all the vertices). Since K is claw-clique, there is a path $P_u \in \mathcal{P}_K$ covering the base of the claw. Thus, $G[v, v', u, t]$ induces a diamond, a contradiction. $\square$

An *independent set* of vertices is a set of vertices no two of which are adjacent. A graph G is said to be *Bipartite* if its set of vertices can be partitioned into two distinct independent sets. There are Bipartite graphs that are non B_1 -EPG, for instance $K_{2,5}$ and $K_{3,3}$ (see [Cohen et al., 2014]). Clearly, since bipartite graphs are triangle-free, any B_1 -EPG representation of a bipartite graph is also a Helly B_1 -EPG representation. A similar result (but a bit weaker) is obtained as corollary of the previous theorem.

Corollary 8. *If G is a Bipartite B_1 -EPG graph then G is a Helly B_1 -EPG graph.*

Proof. The Bipartite graphs are diamond-free, thus by Theorem 7 these graphs are Helly B_1 -EPG graphs. $\square$

A *Block graph* or *Clique Tree* is a type of graph in which every biconnected component (block) is a clique.

Corollary 9. *Block graphs are Helly B_1 -EPG.*

Proof. Block graphs are known to be exactly the chordal diamond-free graphs, so by Theorem 19 of [Ries, 2009], all Block graphs are B_1 -EPG. It follows from Theorem 7 that all Block graphs are Helly B_1 -EPG. $\square$

A *Cactus* (sometimes called a cactus tree) graph is a connected graph in which any two cycles have at most one vertex in common. Equivalently, it is a connected graph in which every edge belongs to at most one cycle, or (for nontrivial cactus) in which every block (maximal subgraph without a cut-vertex) is an edge or a cycle. The family of graphs in which each component is a cactus is closed under graph minor operations. This graph family may be characterized by a single forbidden minor, the diamond graph.

Corollary 10. *Cactus graphs are Helly B_1 -EPG.*

Proof. In [Cela e Gaar, 2019], it is proved that every Cactus graph is a monotonic B_1 -EPG graph (there is a B_1 -EPG representation where all paths are ascending in rows and columns). Thus, Cactus graphs are B_1 -EPG graphs. Since Cactus are diamond-free, by Theorem 7, the proof follows. $\square$

Given a graph G , its *Line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint (i.e. “are incident”) in G . A graph G is a *Line graph of a Bipartite graph* (or simply *Line of Bipartite*) if and only if it contains no claw, no odd cycle, and no diamond as induced subgraph, [Harary e Holzmam, 1974].

Corollary 11. *Line of Bipartite graphs are Helly B_1 -EPG.*

Proof. Line of Bipartite graphs were proved to be B_1 -EPG in [Golumbic et al., 2018]. Since they are diamond-free, the proof follows from Theorem 7. $\square$

The diagram of Figure 7 illustrates the containment relationship between the graph classes studied so far in this work. We list in Figure 8 examples of graphs in each numbered region of the diagram. The numbers of each item below correspond to the regions of the same number in the diagram depicted in Figure 7.

- (1) B_1 -EPG - Helly B_1 -EPG graphs, depicted in Figure 8(a), graph E_1 ;
- (2) Line of Bipartite graphs - Cactus - Block - Bipartite graphs, depicted in Figure 8(b), graph E_2 ;
- (3) Helly B_1 -EPG - Line of Bipartite - Block - Cactus - Bipartite graphs, depicted in Figure 8(c), graph E_3 ;
- (4) Block $\cap$ Line of Bipartite - Cactus - Bipartite, depicted in Figure 8(d), graph E_4 ;
- (5) Block $\cap$ Line of Bipartite $\cap$ Cactus - Bipartite, depicted in Figure 8(e), graph E_5 ;
- (6) Cactus $\cap$ Line of Bipartite - Block - Bipartite. This intersection is empty. Let G be a graph that is Cactus and Line of Bipartite then G is {claw, odd cycle, diamond}-free. But G is not a Bipartite graph, then G has odd cycle. Thus G has at least one triangle, at least one cycle $C_n, n \geq 4$, and G is a connected graph. But given a cycle $C_n, n \geq 4$, if add one vertex any adjacent to this cycle then this induce a claw, absurd with the hypothesis of G is Line of Bipartite;

- (7) Bipartite $\cap$ Line of Bipartite - Cactus - Block graphs, depicted in Figure 8(f), graph E_7 ;
- (8) Bipartite $\cap$ Line of Bipartite $\cap$ Cactus - Block graphs, depicted in Figure 8(g), graph E_8 ;
- (9) Bipartite $\cap$ Line of Bipartite $\cap$ Cactus $\cap$ Block graphs, depicted in Figure 8(h), graph E_9 ;
- (10) Bipartite $\cap$ Cactus $\cap$ Block - Line of Bipartite graphs, depicted in Figure 8(i), graph E_{10} ;
- (11) Bipartite $\cap$ Cactus - Block - Line of Bipartite graphs, depicted in Figure 8(j), graph E_{11} ;
- (12) Bipartite $\cap$ Helly B_1 -EPG - Cactus - Block - Line of Bipartite graphs, depicted in Figure 8(k), graph E_{12} ;
- (13) Bipartite - B_1 -EPG graphs, depicted in Figure 8(l), graph E_{13} ;
- (14) Block - Bipartite - Line of Bipartite - Cactus graphs, depicted in Figure 8(m), graph E_{14} ;
- (15) Block $\cap$ Cactus - Line of Bipartite - Bipartite graphs, depicted in Figure 8(n), graph E_{15} ;
- (16) Cactus - Block - Line of Bipartite - Bipartite graphs, depicted in Figure 8(o), graph E_{16} , the odd cycles C_{2n+1} , $n \geq 2$;
- (17) Helly EPG - B_1 -EPG - Bipartite graphs, depicted in Figure 8(p), graph E_{17} ;

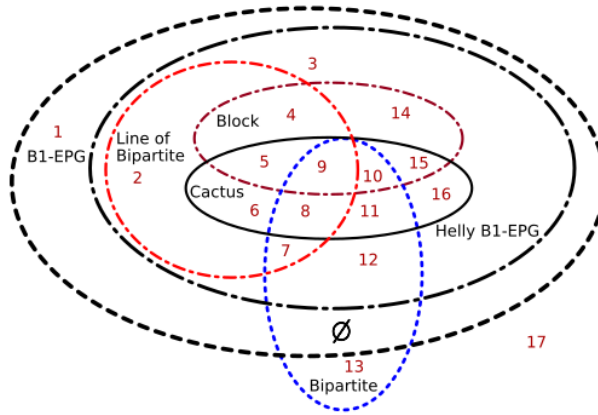


Figure 7: Diagram of some graph classes.

4. Concluding Remarks

We have determined that the Block, Cactus, Line of Bipartite and the Bipartite (that are B_1 -EPG) graph classes are contained in Helly B_1 -EPG. In addition we show that every B_1 -EPG and diamond-free graph is a Helly B_1 -EPG graph. Furthermore, we define a set of subgraphs, namely $\{S_3, S_{3'}, S_{3''}, C_4\}$, that are present in any B_1 -EPG graph that is not Helly. We present a diagram of relationship between these classes with some class divider graph in instance.

Thus this paper presents graph classes that are properly contained in Helly B_1 -EPG and other related structures. We leave three questions to be investigated, in relation to the presented results.

1. If we take a fixed k greater than 1, in some B_k -EPG representation, which graph classes are contained and have the Helly property?

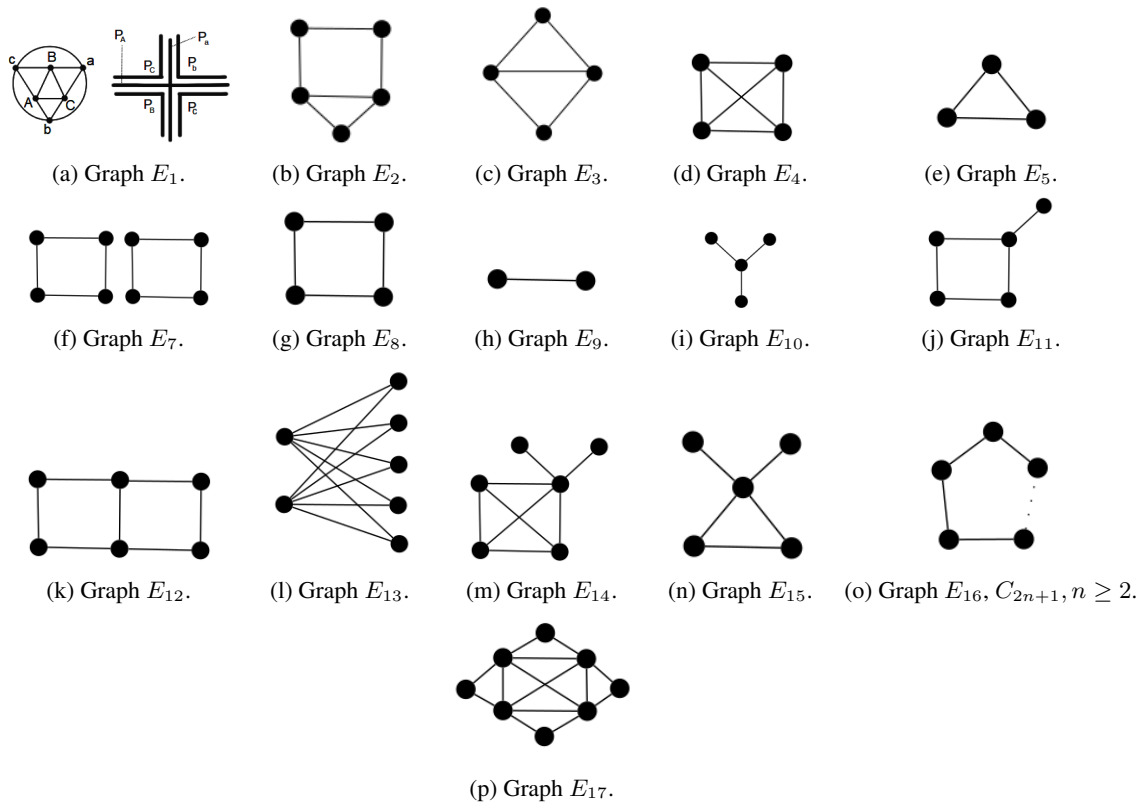


Figure 8: The set of instances for Venn Diagram of the graph classes of this paper.

2. Within the class of graphs B_1 -EPG itself, if we consider classes that were not covered by this paper, such as Weakly Cordal graphs, Hereditary Distance, Cographs, Split, etc. Which graphs of these classes have the Helly property?
3. The B_1 -EPG and Helly B_1 -EPG are hereditary classes. Recognizing B_1 -EPG and Helly B_1 -EPG graphs is hard. But, it must be possible to characterize, even if by an infinite family of subgraphs, all graphs B_1 -EPG and Helly B_1 -EPG. What is this family of graphs?

Acknowledgement: The third author (Tanilson Dias) would like to thank the partial financing of this study by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

References

- Bornstein, C. F., Golumbic, M. C., Santos, T. D., Souza, U. S., e Szwarcfiter, J. L. (2019). The complexity of Helly B_1 -EPG graph recognition. *arXiv preprint arXiv:1906.11185*.
- Cela, E. e Gaar, E. (2019). Monotonic representations of outerplanar graphs as edge intersection graphs of paths on a grid. *arXiv preprint arXiv:1908.01981*.
- Cohen, E., Golumbic, M. C., e Ries, B. (2014). Characterizations of cographs as intersection graphs of paths on a grid. *Discrete Applied Mathematics*, 178:46–57.

- Golumbic, M., Lipshteyn, M., e Stern, M. (2009). Edge intersection graphs of single bend paths on a grid. *Networks*, 54:130–138.
- Golumbic, M. C., Morgenstern, G., e Rajendraprasad, D. (2018). Edge-intersection graphs of boundary-generated paths in a grid. *Discrete Applied Mathematics*, 236:214–222.
- Harary, F. e Holzmam, C. (1974). Line graphs of bipartite graphs. *Revista de La Sociedad Matemática de Chile*, 1(1):19–22.
- Ries, B. (2009). Some properties of edge intersection graphs of single bend paths on a grid. *Electronic Notes in Discrete Mathematics*, 34:29–33.