Sampled-Data Control of Interval Systems with Guaranteed LQ/ $\mathcal{H}_2$ Performance

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Abstract: Robustness is one of the many important properties a control system should present, together with its performance. In this paper, we consider linear systems with interval uncertainties and propose design techniques for robust controllers that ensure an upper bound for an associated quadratic cost. These results are also extended for the sampled-data case and can deal with time-varying uncertainties. Numerical examples point out the main features of the proposed methods.

Resumo: Robustez é uma das muitas propriedades que um sistema de controle deve apresentar, juntamente com desempenho. Neste artigo, o objetivo é controlar um sistema linear com incertezas intervalares usando um sinal de controle amostrado a fim de minimizar um critério de desempenho quadrático. As condições desenvolvidas permanecem válidas mesmo para sistemas com incertezas variantes no tempo. Por fim, apresentamos exemplos numéricos para apontar as principais características dos métodos propostos.

Keywords: Interval systems; sampled control; linear quadratic regulator; quadratic linear methods; quadratic stability; robust controller.

Palavras-chaves: Sistemas intervalares; controle amostrado; regulador linear quadrático; métodos lineares quadráticos; estabilidade quadrática; controlador robusto.

1. INTRODUCTION

Linear quadratic (LQ) methods are one of the most useful optimal control tools for practical engineering (Anderson and Moore, 2007). Assigning a quadratic cost to a linear control problem is natural in many applications and, thus, it has become a standard approach for many control practitioners. The final aim of this paper is to control a linear uncertain system of the form

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t), \quad (1)$$

in which $\mathbf{A} \in [\underline{\mathbf{A}}, \overline{\mathbf{A}}]$ and $\mathbf{B} \in [\underline{\mathbf{B}}, \overline{\mathbf{B}}]$, with the sampled-data state-feedback control signal

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \quad (2)$$

while aiming at minimizing an upper bound for the quadratic cost

$$J(u) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt, \quad (3)$$

for some given matrices $Q \succeq 0$ and $R \succ 0$. To that end, we must review the literature on interval systems and adapt classic results to cope with the quadratic cost optimization.

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1.1 Related Work

The literature is abundant in the area of robust control of interval and (the closely related) polytopic systems. Following the remarkable robust stability result provided by Kharitonov’s Theorem (Kharitonov, 1978; Ackermann, 1993), classic references in the area of interval systems (Sezer and Šiljak, 1994; Mansour, 1989; Delgado-Romero et al., 1997; Daoyi, 1985) focused on devising simple tests and conditions for the stability of an interval dynamic system. Stability conditions for interval systems have been revisited and restated in the linear matrix inequality (LMI) (Boyd et al., 1994) framework in Mao and Chu (2003); Zhang et al. (2006), in which other properties such as controllability are also investigated; these new results are based on Petersen’s Lemma (Petersen, 1987; Shcherbakov and Topunov, 2008; Ji and Su, 2016) for robust stability under norm-bounded uncertainties. These LMI conditions provide a simple and computationally efficient stability test for the interval system and can be extended to design stabilizing controllers, for instance. Even though these conditions were initially published as necessary and sufficient for quadratic stability and stabilizability of an uncertain system, it was later verified that they are, in fact, only sufficient; the necessity implication of Petersen’s Lemma does not hold for more than one norm-bounded uncertainty (Shcherbakov and Topunov, 2008; Ji and Su, 2016; Yang and Lum, 2005; Mao and Chu, 2006). In the

following papers Hong et al. (2006); Mao (2004); Lee et al. (2006) robust conditions $\mathcal{H}_\infty/\mathcal{H}_2$ have been proposed in terms of LMIs for continuous interval systems, as the results are based on the Petersen Lemma these conditions are only sufficient.

1.2 Contributions of this Paper

In this paper, we review the robust stability and stabilization conditions presented in Zhang et al. (2006) and extend these results to consider a guaranteed quadratic cost that must be minimized, as in an LQ/ $\mathcal{H}_2$ fashion. It is important to state that this robust control problem has already been addressed in the literature using polytopic models and affine Lyapunov functions. The main drawback presented by this approach is the exponential number of vertices: an interval matrix with m uncertain (independent) entries can be represented by a matrix polytope with 2^m vertices, which means more constraints and variables are required to assess its stability. Our conditions, albeit more conservative, only need one constraint and a polynomial number of variables and, thus, they can be solved more efficiently even for large systems. Moreover, as it will be clear in the sequence, our conditions can be promptly extended to consider time-varying uncertainties. For brevity and to simplify notation, we present mainly time-invariant results, but also illustrate the varying case in an example. Finally, these design conditions are applied to the design of sampled-data controllers.

2. MATHEMATICAL PRELIMINARIES

2.1 Notation

We now introduce some notation that shall be used throughout the paper. The sets of natural, real and non-negative real numbers are indicated by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$, respectively. The set of time instants is denoted by $\mathbb{T}$ and corresponds to $\mathbb{R}_+$ for continuous-time signals and to $\mathbb{N}$ in the discrete case. The set of real $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$ and the set of n -dimensional real column vectors is denoted by $\mathbb{R}^n$. For any matrix $X \in \mathbb{R}^{m \times n}$, $X^T \in \mathbb{R}^{n \times m}$ denotes its transpose. Additionally, for any matrix $X = (x_{ij}) \in \mathbb{R}^{m \times n}$, $X > 0$ ($X \geq 0$) denotes that $x_{ij} > 0$ ($x_{ij} \geq 0$) for all i, j . For any symmetric matrix $X = X^T \in \mathbb{R}^{n \times n}$, $X \succ 0$ ($X \succeq 0$) denotes that X is positive definite (semidefinite). Now, we define $\mathbf{He}(X)$ as $\mathbf{He}(X) := X + X^T$. The set $\mathbb{S}_+^n$ is formed by all symmetric positive definite matrices of order in $\mathbb{R}^{n \times n}$. For any two given matrices $\underline{X}, \bar{X} \in \mathbb{R}^{m \times n}$ such that $\underline{X} \leq \bar{X}$, we define the interval matrix $[\underline{X}, \bar{X}]$ as the set

$$[X] = [\underline{X}, \bar{X}] := \left\{ X \in \mathbb{R}^{m \times n} : \underline{X} \leq X \leq \bar{X} \right\}, \quad (4)$$

whose *center* and *radius* are defined as $X_0 := \frac{1}{2}\underline{X} + \frac{1}{2}\bar{X}$ and as $\Delta X := \frac{1}{2}\bar{X} - \frac{1}{2}\underline{X}$, respectively. We may also write the matrix interval above as

$$[X] := \left\{ X_0 + \sum_{i,j} e_i \delta x_{ij} f_j^T : |\delta x_{ij}| \leq \Delta x_{ij} \right\}, \quad (5)$$

in which $\Delta x_{ij} = [\Delta X]_{ij}$ and e_i and f_j are the i -th and the j -th columns of compatible identity matrices. Uncertain or arbitrary elements in a matrix interval are denoted in bold. Finally, to optimize the space in the description of

symmetric matrices, we use the symbol $\star$ to denote a block whose symmetric correspondent is already described.

2.2 Interval Systems

Consider the following linear time-invariant system

$$\Sigma_d : \begin{cases} x[k+1] = \mathbf{A}x[k] + \mathbf{B}u[k], & x[0] = x_0, \\ y[k] = \mathbf{C}x[k] + \mathbf{D}u[k], \end{cases} \quad (6)$$

where $x : \mathbb{T} \rightarrow \mathbb{R}^{n_x}$ is the state, $u : \mathbb{T} \rightarrow \mathbb{R}^{n_u}$ is the control input and $y : \mathbb{T} \rightarrow \mathbb{R}^{n_y}$ is the output. In this article, Σ_d is said to be *interval systems*, meaning that their realization matrices $\mathbf{A} \in [A]$, $\mathbf{B} \in [B]$, $\mathbf{C} \in [C]$ and $\mathbf{D} \in [D]$ are not precisely known. For simplicity, define the set

$$\mathbb{X} = [A] \times [B] \times [C] \times [D], \quad (7)$$

which allows us to write the uncertainties in compact form $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in \mathbb{X}$.

2.3 Petersen's Lemma on Quadratic Stability

In this section, we present some relevant auxiliary results for the main theoretical developments of this article. We begin by the important matrix lemma stated by Ian Petersen:

Lemma 1. (Petersen, 1987; Shcherbakov and Topunov, 2008) Let $Y \in \mathbb{S}^n$, $M \in \mathbb{R}^{n \times p}$, and $N \in \mathbb{R}^{q \times n}$ be given. The inequality

$$Y + M\Delta[k]N + N^T\Delta[k]^T M^T \prec 0, \quad \forall k \in \mathbb{N}, \quad (8)$$

holds for all $\Delta : \mathbb{N} \rightarrow \mathbb{R}^{p \times q}$ such that $\Delta[k]^T \Delta[k] \preceq I$ for all $k \in \mathbb{N}$ if, and only if, there exists $\epsilon > 0$ such that

$$Y + \epsilon MM^T + \epsilon^{-1} N^T N \prec 0. \quad (9)$$

Unfortunately, this remarkable result cannot be extended for several uncertain matrices without losing necessity; see Ji and Su (2016) for a generalized necessary and sufficient result for two uncertain matrices. The same drawback exists in the S-procedure and, as there is a relationship between Petersen's Lemma and the S-procedure, this might be one way of justifying this fact (Shcherbakov and Topunov, 2008). As a consequence of this loss of necessity, the results developed Mao and Chu (2003) and Zhang et al. (2006), which made use of Petersen's Lemma for interval systems, are not necessary, as pointed out by Yang and Lum (2005). Nevertheless, the conditions presented in both papers are very interesting as they are based on only one linear matrix inequality, which means they can tackle even systems with a relatively large number of uncertain parameters. As pointed out by Shcherbakov and Topunov (2008), the conservativeness of such conditions is acceptable in practice.

Remark 1. As our main results are heavily dependent on Petersen's Lemma, the same design conditions presented in this paper can deal with interval time-varying uncertainties with no additional assumptions.

3. LINEAR QUADRATIC ROBUST CONTROL OF DISCRETE-TIME INTERVAL SYSTEMS

In this section, we extend the robust control results presented in Zhang et al. (2006) for discrete-time systems. We aim to include a performance bound in the design

techniques, which means that not only stability is sought, but also some desired guaranteed quadratic cost. Hence, we seek to stabilize Σ_d by means of a robust state-feedback control law that minimizes an upper bound for

$$J(u) = \sum_{k \in \mathbb{N}} \|y[k]\|^2. \quad (10)$$

This cost is closely related to the LQR performance index; therefore, this problem can be easily solved in the precisely known case by means of an algebraic Riccati equation Anderson and Moore (2007). As our problem involves an interval uncertain system, we begin by stating the following lemma.

Lemma 2. Let Σ_d be an interval system with $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in \mathbb{X}$. There exist a quadratically stabilizing feedback gain $K \in \mathbb{R}^{n_u \times n_x}$ for Σ_d and a positive scalar $\mu > 0$ such that the quadratic cost defined in (10) verifies the bound

$$J(u) < \mu \quad (11)$$

for all $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in \mathbb{X}$ if, and only if, there exist matrices $X \in \mathbb{S}_+^{n_x}$, $G \in \mathbb{R}^{n_x \times n_x}$ and $L \in \mathbb{R}^{n_u \times n_x}$ such that

$$\begin{bmatrix} \mathbf{He}(G) - X & \star & \star \\ \mathbf{A}G + \mathbf{B}L & X & \star \\ \mathbf{C}G + \mathbf{D}L & 0 & I \end{bmatrix} \succ 0 \quad (12)$$

holds for all $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in \mathbb{X}$. Furthermore, the quadratically stabilizing state-feedback gain is given by $K = LG^{-1}$ and the upper bound for the quadratic cost is $\mu := x_0^T X^{-1} x_0$, where x_0 is the initial condition.

Proof. 1. The proof can be found in the following paper De Oliveira et al. (2002).

The conditions stated in this lemma involve linear matrix inequalities that depend on all realization matrices in $\mathbb{X}$. The following theorem provides a set of computationally viable conditions for the robust control design problem.

Theorem 1. Let Σ_d be an interval system with $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in \mathbb{X}$ and let the scalar $\mu > 0$ be given. There exists a quadratically stabilizing feedback gain $K \in \mathbb{R}^{n_u \times n_x}$ for Σ_d such that the quadratic cost defined in (10) verifies the bound

$$J(u) < \mu \quad (13)$$

for all $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in \mathbb{X}$ if there exists matrices $X \in \mathbb{S}_+^{n_x}$, $G \in \mathbb{R}^{n_x \times n_x}$ and $L \in \mathbb{R}^{n_u \times n_x}$ and positive scalars α_{ij} , $i, j \in \{1, \dots, n_x\}$, β_{ij} , $i \in \{1, \dots, n_x\}$ and $j \in \{1, \dots, n_u\}$, γ_{ij} , $i \in \{1, \dots, n_y\}$ and $j \in \{1, \dots, n_x\}$, and ε_{ij} , $i \in \{1, \dots, n_y\}$ and $j \in \{1, \dots, n_u\}$, such that the linear matrix inequalities

$$\begin{bmatrix} \mu & \star \\ x_0 & X \end{bmatrix} \succ 0, \quad (14)$$

$$\begin{bmatrix} \mathbf{He}(G) - X & \star & \star & \star & \star & \star \\ A_0 G + B_0 L & X - S & \star & \star & \star & \star \\ C_0 G + D_0 L & 0 & I - T & \star & \star & \star \\ \mathcal{G}_a & 0 & 0 & \mathcal{A} & \star & \star \\ \mathcal{G}_c & 0 & 0 & 0 & \Gamma & \star \\ \mathcal{L}_b & 0 & 0 & 0 & 0 & \mathcal{B} \\ \mathcal{L}_d & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \succ 0, \quad (15)$$

in which

$$S = \sum_{i,j} \Delta a_{ij}^2 \alpha_{ij} e_i e_i^T + \sum_{i,j} \Delta b_{ij}^2 \beta_{ij} e_i e_i^T, \quad (16)$$

$$T = \sum_{i,j} \Delta c_{ij}^2 \gamma_{ij} g_i g_i^T + \sum_{i,j} \Delta d_{ij}^2 \varepsilon_{ij} g_i g_i^T, \quad (17)$$

$$\mathcal{G}_a = \underbrace{[G^T \dots G^T]^T}_{n_x}, \quad \mathcal{G}_c = \underbrace{[G^T \dots G^T]^T}_{n_y}, \quad (18)$$

$$\mathcal{L}_b = \underbrace{[L^T \dots L^T]^T}_{n_x}, \quad \mathcal{L}_d = \underbrace{[L^T \dots L^T]^T}_{n_y}, \quad (19)$$

$$\mathcal{A} = \mathbf{diag}(\alpha_{11}, \dots, \alpha_{1n_x}, \dots, \alpha_{n_x 1}, \dots, \alpha_{n_x n_x}), \quad (20)$$

$$\mathcal{G} = \mathbf{diag}(\gamma_{11}, \dots, \gamma_{1n_x}, \dots, \gamma_{n_y 1}, \dots, \gamma_{n_y n_x}), \quad (21)$$

$$\mathcal{B} = \mathbf{diag}(\beta_{11}, \dots, \beta_{1n_u}, \dots, \beta_{n_x 1}, \dots, \beta_{n_x n_u}), \quad (22)$$

$$\mathcal{E} = \mathbf{diag}(\varepsilon_{11}, \dots, \varepsilon_{1n_u}, \dots, \varepsilon_{n_y 1}, \dots, \varepsilon_{n_y n_u}), \quad (23)$$

hold; vectors e_i and g_i are the i -th columns of identity matrices of compatible dimensions. The matrices A_0 , B_0 , C_0 and D_0 are the center matrices of $\mathbb{X}$. The radius Δa_{ij} , Δb_{ij} , Δc_{ij} and Δd_{ij} are defined as $\Delta X := \frac{1}{2} \bar{X} - \frac{1}{2} \underline{X}$ where X is respectively $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ for each radius. Furthermore, the quadratically stabilizing state-feedback gain is given by $K = LG^{-1}$.

Proof. 2. First, we apply Schur Complement and some algebraic manipulations on (14). By Petersen's Lemma (Lemma 1), we can rewrite it in interval form as (12), implying that $u = Kx$, with $K = LG^{-1}$ makes Σ_d quadratically stable and verifies the guaranteed performance $J(u) < \mu = x_0^T X^{-1} x_0$. For more details, the papers Mao and Chu (2003); Zhang et al. (2006) present methods similar to those used in this theorem.

Example 1. To illustrate the results related to the LQ problem in discrete time we consider the example given in Zhang et al. (2006). In this case, we take the discrete-time uncertain system Σ_d with

$$[A] = \begin{bmatrix} [0.5000 & 1.0000] & [0.2499 & 0.5000] \\ [0.2499 & 0.5000] & [0.1666 & 0.3333] \\ [0.1666 & 0.3333] & [0.1249 & 0.2501] \\ [0.1249 & 0.2501] & [0.1000 & 0.2001] \end{bmatrix},$$

$$\begin{bmatrix} [0.1666 & 0.3333] & [0.1249 & 0.2501] \\ [0.1249 & 0.2501] & [0.1000 & 0.2001] \\ [0.1000 & 0.2001] & [0.0832 & 0.1677] \\ [0.0832 & 0.1667] & [0.0713 & 0.1430] \end{bmatrix},$$

$$[B] = \begin{bmatrix} [1.0000 & 1.5000] \\ [1.0000 & 1.5000] \\ [1.0000 & 1.5000] \\ [1.0000 & 1.5000] \end{bmatrix}.$$

Realisation matrices C and D and initial conditions x_0 were chosen according to:

$$\underline{C} = \bar{C} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \underline{D} = \bar{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

When we apply the conditions of the Theorem 1, we get the robust state feedback gain

$$K = [-0.3829 \quad -0.2216 \quad -0.1589 \quad -0.1246].$$

To devise this feedback gain, 63 inequality lines and 76 variables were necessary. The overall computation time was of 2.8018s. All the code in this paper was run on a

Tabela 1. Comparative table of closed-loop performance achieved by two design techniques.

	Central	Interval
Guaranteed Cost	—	11.4158
Worst Case	12.1954	8.8749
Average Cost	5.8222	6.0017
Std. Deviation	1.0934	0.6961

Dell computer with Windows 10 64 bits operating system, Intel Xeon processor X5570, 2.93 GHz, and 12GB memory. Also, interval operations and functions are carried out with INTLAB Rump (1999), which is a computational package developed for reliable computing that runs on MATLAB.

The robust controller provided by the polytopic approach presented in de Oliveira et al. (1999) is a natural candidate for comparison. However, due to the combinatorial nature of this design technique, which requires around a million linear matrix inequalities, the optimization procedure did not finish within 72 hours of computation. We also compute the central LQR controller, by using the nominal matrices A_0 and B_0 and a discrete-time Riccati equation. This nominal gain is given by

$$K = [-0.3001 \quad -0.1799 \quad -0.1315 \quad -0.1045].$$

These controllers were validated by a Monte Carlo procedure. In this example, we generate around 100 thousand possible realization matrices and compute the closed-loop performance. These simulation results are compiled in Table 1. The robust control approach presents a best worst-case cost than the nominal controller.

Example 2. Now we illustrate a system with a time-varying uncertainty. Let us consider a discrete-time system

$$\Sigma_d : \begin{cases} x[k+1] = \mathbf{A}[k]x[k] + \mathbf{B}u[k], x[0] = x_0, \\ y[k] = \mathbf{C}x[k] + \mathbf{D}u[k]. \end{cases}$$

with

$$\mathbf{A}[k] \in \left[\begin{bmatrix} 0.3 & 0.5 \\ -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 0.9 \end{bmatrix} \right], \underline{B} = \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\underline{C} = \overline{C} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \underline{D} = \overline{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For this system, we design a robust controller using the interval approach presented in Theorem 1 and obtain

$$K = [-0.2084 \quad -0.9626],$$

which ensures $J(u) < 6.4174$. Additionally, we also design a robust controller using the approach presented in (de Oliveira et al., 1999), which may not be valid for time-varying systems. This approach yields the robust gain

$$K = [-0.2796 \quad -0.9367],$$

with guaranteed cost $J(u) < 6.0343$; remember that this cost only holds for time-invariant uncertainties.

We simulate both closed-loop systems with a periodic $A[k]$, for which $A[k] = A[k \bmod 5]$ and

$$\begin{aligned} A[0] &= \begin{bmatrix} 0.5 & 0.7 \\ -0.2 & 0.7 \end{bmatrix}, & A[1] &= \begin{bmatrix} 0.3 & 0.7 \\ -0.2 & 0.9 \end{bmatrix}, \\ A[2] &= \begin{bmatrix} 0.5 & 0.7 \\ 0.4 & 0.9 \end{bmatrix}, & A[3] &= \begin{bmatrix} 0.5 & 0.5 \\ -0.2 & 0.9 \end{bmatrix}, \\ A[4] &= \begin{bmatrix} 0.3 & 0.7 \\ -0.2 & 0.9 \end{bmatrix}. \end{aligned}$$

For this case, our controller yields a closed-loop cost of 6.0656 and the polytopic approach provides a cost of 6.5725. This illustrates that our results can be used even when the uncertainty varies with time.

4. APPLICATION: SAMPLED-DATA LINEAR QUADRATIC ROBUST CONTROL

Let us now briefly discuss how the main ideas devised in this paper can be applied to design sampled-data state-feedback robust controllers for sampled-data systems. Our aim is to solve the problem presented in the Introduction of this paper, which focuses on devising a control law u of the form (2) that ensures robust stability to (1) whilst simultaneously minimizing an upper bound for (3).

Our first step is to obtain a discrete-time interval realization for (1), we use the classic choice

$$\begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} = e^{h\mathcal{A}}, \text{ with } \mathcal{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix},$$

note that $\mathcal{A} \in \mathbb{R}^{(n_x+n_u) \times (n_x+n_u)}$. To compute the exponential, we performed the scaling and squaring method presented in (Goldsztejn and Neumaier, 2014) to compute intervals $[A_d]$ and $[B_d]$ such that

$$e^{h\mathcal{A}} = \exp \left(h \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) \in \begin{bmatrix} [A_d] & [B_d] \\ 0 & I \end{bmatrix}. \quad (24)$$

These computations, thus, yield the discrete-time interval system

$$x[k+1] = \mathbf{A}_d x[k] + \mathbf{B}_d u[k], \quad x[0] = x_0, \quad (25)$$

in which $\mathbf{A}_d \in [A_d]$ and $\mathbf{B}_d \in [B_d]$. The associated equivalent quadratic cost can be devised using

$$\begin{bmatrix} Q_d & S_d \\ \star & R_d \end{bmatrix} = \int_0^h e^{\tau\mathcal{A}^\top} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} e^{\tau\mathcal{A}} d\tau.$$

Indeed, using a self-validating numeric quadrature procedure presented in (Moore et al., 2009), we compute intervals $[Q_d]$, $[S_d]$ and $[R_d]$ such that

$$\int_0^h e^{\tau\mathcal{A}^\top} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} e^{\tau\mathcal{A}} d\tau \in \begin{bmatrix} [Q_d] & [S_d] \\ \star & [R_d] \end{bmatrix}. \quad (26)$$

After computing the matrices of the discretized equivalent system, we compute a robust sampled-data controller (2) that optimizes (3) for (1) using Theorem 1 for the equivalent system. Note that these techniques are based on well established numerical procedures and, therefore, can be efficiently computed. Then, using Cholesky's factorization (Horn and Johnson, 2012), we obtain the matrices $[C_d]$ and $[D_d]$. A singular value decomposition (SVD) based decomposition (Horn and Johnson, 2012), can also be used whenever the Cholesky computation fails due to singularity of some matrix in (26).

Example 3. Consider the uncertain system (1) with realization matrices taken from Example 2 of Wang and Michel (1994):

$$[A] = \begin{bmatrix} 1 \pm 0.1 & -1 \pm 0.1 \\ 0 \pm 0.5 & 4 \pm 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1.0000 \\ -1.0000 \end{bmatrix}.$$

Our goal is to optimize the quadratic cost (3), with

$$Q = I, \quad R = 1,$$

using a sampled-data control law of the form (2) with sampling period of $h = 0.1$ s. We assume the system evolves from $x_0 = [1 \ 1]^T$.

We begin by computing the intervals $[A_d]$ and $[B_d]$ using the scaling and squaring method, with both parameters set to 10, obtaining:

$$[A_d] = \begin{bmatrix} 1.0937 & 1.1167 \\ -0.0066 & 0.0066 \end{bmatrix} \begin{bmatrix} -0.1440 & -0.1138 \\ 1.4613 & 1.5224 \end{bmatrix},$$

$$[B_d] = \begin{bmatrix} 0.1098 & 0.1124 \\ -0.1246 & -0.1213 \end{bmatrix}.$$

We also compute the intervals $[C_d]$ and $[D_d]$ following the procedure discussed above:

$$[C_d] = \begin{bmatrix} 0.3309 & 0.3346 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix} \begin{bmatrix} -0.0228 & -0.0154 \\ 0.3864 & 0.3966 \\ 0.0000 & 0.0000 \end{bmatrix},$$

$$[D_d] = \begin{bmatrix} 0.0167 & 0.0177 \\ -0.0207 & -0.0185 \\ 0.3163 & 0.3167 \end{bmatrix}.$$

These realization matrices yield the discrete-time equivalent system.

Using the design conditions of Theorem 1 with the equivalent system, we obtain the robust gain

$$K = [8.3951 \ 18.8406].$$

This design procedure required 25 lines of inequalities and 25 optimization variables and took 0.4482s to terminate. We use the conditions in de Oliveira et al. (1999) to design another robust gain, which is given by

$$K = [6.6588 \ 15.4701].$$

This alternative method required 40,960 lines of inequalities and 12,295 decision variables to run and took 146.5709s to finish. In this example, the central controller is obtained by solving the sampled-data LQR problem for the central continuous-time matrices A_0 and B_0 , following Souza et al. (2014). This central gain is given by

$$K = [3.6284 \ 11.8497].$$

These sampled-data controllers were also validated by a Monte Carlo procedure. In this example, we generate around 55 thousand possible realization matrices and compute the closed-loop performance. These simulation results are compiled in Table 2. Once again, the central controller presents a good average performance but is poor when robustness is considered. In terms of worst case and of average performance, the interval-based controller is slightly – around 10% in the worst case – outperformed by the polytopic-based controller, as expected, since the interval-based design can cope even with time varying uncertainties. Note that this small sampled-data design example already shows that the interval-based design procedure can deal with small number of resources whereas the polytopic-based design method used a large number of variables and constraints. Simulation results for the central, interval, and polytopic controllers are shown in Figures 1, 2 and 3, respectively.

Tabela 2. Comparative table of closed-loop performance achieved by three design techniques.

	Central	Interval	Polytopic
Guaranteed Cost	–	271.7448	191.7128
Worst Case	382.8590	180.9792	164.3040
Average Cost	103.8698	154.9034	138.0867
Std. Deviation	20.2902	4.5211	5.2506

Figura 1. Closed-loop response with the central controller.

Figura 2. Closed-loop response with the interval-based controller.

Figura 3. Closed-loop response with the polytopic-based controller.

5. CONCLUSION

In this paper, we provide novel robust control design techniques for interval systems. We present results that ensure

stability and guaranteed LQ (or $\mathcal{H}_2$) performance discrete-time interval systems. As an application, we consider a linear quadratic sampled-data control design problem for interval systems. As the numerical examples point out, the design techniques presented in this paper can deal with systems that present a relatively large number of uncertain parameters and interval time-varying uncertainties, which cannot be tackled by polytopic-based approaches.

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