Robust $\mathcal{D}$ -Stabilisation of Uncertain Systems via Static Output Feedback ^{*}

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Abstract: In this paper, a robust static output feedback (SOF) $\mathcal{D}$ -stabilisation design strategy for uncertain linear time-invariant (LTI) systems is proposed. The SOF controller design is based on a two-stage method and the robust static gains are obtained by solving a set of linear matrix inequalities (LMI) conditions. The $\mathcal{D}$ -region considered for the eigenvalue placement is a circle in the complex left half-plane which is defined according to two scalar parameters. We also propose more relaxed conditions for the SOF controller design, when compared to the conservative preliminary LMI constraints, by considering parameter-dependent Lyapunov functions (PDLF) and parameter-dependent Finsler's Lemma additional variables (PDFV). Some academic examples are presented in order to attest for the validity of the proposed results, followed by a practical implementation on an active suspension system, assuming that the feedback loop is composed by only one of its four system state variables.

Keywords: Static output feedback, Robust pole placement, Linear time-invariant systems, Relaxation strategies, Practical implementation.

1. INTRODUCTION

Dealing with robust stabilisation of uncertain systems is a rather important topic in control design literature, as it has received considerable attention in the past decades (Chesi, 2017). This fact is justified due to the practical issue of not having precise information about one or more of the system parameters and the need for ensuring stability for all possible uncertainties (Chesi, 2017).

However, further from guaranteeing the asymptotic stability, in several practical control problems, it is also necessary to provide improved dynamic performance to the system closed-loop response (Chilali et al., 1999). For instance, the settling time and steady-state signal overshoot are examples of indexes related to the system performance, which can be optimised with an appropriately controller design (dos Santos et al., 2018).

A strategy for enhancing the system dynamic characteristics, based on the specification of desired values for such transient parameters, is to properly enforce a robust closed-loop pole placement in a particular region of the complex left half-plane, namely $\mathcal{D}$ -stabilisation (Chilali and Gahinet, 1996). This technique has been investigated and associated to a series of stability analysis and control strategies, such as the pole placement in regions described in terms of linear matrix inequalities (LMI) with $\mathcal{H}_\infty$ design (Chilali et al., 1999), robust $\mathcal{D}$ -stability analysis via polynomial-type multipliers (Ebihara et al., 2005), and

also in output feedback perspective (Chesi, 2017; dos Santos et al., 2018), for mentioning a few. Moreover, regarding the definition of a circular $\mathcal{D}$ -region for pole placement of uncertain linear time-invariant (LTI) systems, one may find works related to state feedback (Leite et al., 2004) and state derivative feedback (Beteto et al., 2018; da Silva et al., 2012), for example.

In this work, we focus on the static output feedback (SOF) control, whose relevance in practical applications and the challenging nature of its mathematical description (Trofino and Kucera, 1993) has served as motivation for the development of a vast number of research papers addressing this topic. For example, one may found contributions in non-linear systems, as the discrete-time fuzzy T-S design strategy for SOF controllers presented in Chadli and Guerra (2012), for uncertain LTI systems as the continuous-time robust SOF control design proposed in Sereni et al. (2018), and several other applications as the reader may see in the survey papers Sadabadi and Peaucelle (2016), Syrmos et al. (1997), and references therein.

With this scope, we proposed a novel LMI-based strategy for robust pole placement in SOF control design, considering the $\mathcal{D}$ -region for pole placement as a circle in the complex left half-plane. The SOF controller synthesis is based on the two-stage method (Sereni et al., 2018; Agulhari et al., 2012) and described in terms of LMI, which has a consolidated background in system analysis and controller synthesis (Ebihara et al., 2005; Boyd et al., 1994). We also propose less conservative LMI conditions for the robust SOF controller synthesis, based on the use of parameter-dependent Lyapunov functions (PDLF) (Oliveira and Peres, 2007; De Oliveira et al., 1999) and parameter-dependent Finsler's Lemma addi-

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tional variables (PDFV) (Sereni et al., 2018). The proposed method has its efficacy attested by the presentation of computational examples and feasibility analysis for a conservativeness study comparison. Additionally, the results of a practical implementation of a robust SOF controller designed with our $\mathcal{D}$ -stabilisation method for an active suspension system is presented, corroborating the practical applicability of the proposed strategy.

Throughout the text, we use the following notation: $M > 0$ ($M < 0$) implies that M is a positive definite (negative definite) matrix; $(\cdot)'$ denotes the transpose of a matrix; $\mathbb{R}$ denotes the set of the real numbers; $\text{He}(M) = M + M'$; $(*)$ denotes the symmetric block in a partitioned symmetric matrix; and, for a compact representation, M_α means that M is an uncertain matrix $M(\alpha)$ that can be represented in a polytopic domain parametrised in terms of a vector α , and it is defined as

$$M_\alpha = M(\alpha), \text{ where } M(\alpha) = \sum_{i=1}^N \alpha_i M_i, \quad (1)$$

and M_i denotes the i -th polytope vertex, N is the number of vertices of the polytope. The parameters α_i are unknown constants belonging to the unitary simplex set, Δ_N , defined as

$$\Delta_N = \{\alpha \in \mathbb{R}^n : \sum_{i=1}^N \alpha_i = 1; \alpha_i \geq 0; i = 1, \dots, N\}. \quad (2)$$

2. PRELIMINARIES

Consider the uncertain LTI system whose dynamics can be described in terms of the state-space representation

$$\begin{aligned} \dot{x}(t) &= A_\alpha x(t) + B_\alpha u(t) \\ y(t) &= C_\alpha x(t), \end{aligned} \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the measured output vector, and $u(t) \in \mathbb{R}^m$ is the control input vector. Moreover, $A_\alpha \in \mathbb{R}^{n \times n}$, $B_\alpha \in \mathbb{R}^{n \times m}$, and $C_\alpha \in \mathbb{R}^{p \times n}$ are uncertain matrices defined according to (1).

The problem consists in obtaining a control law $u(t) = Ly(t)$, where $L \in \mathbb{R}^{m \times p}$ is a conveniently designed robust SOF gain matrix such that the closed-loop system

$$\dot{x}(t) = (A_\alpha + B_\alpha LC_\alpha)x(t) \quad (4)$$

is asymptotic stable and all its eigenvalues (λ) are allocated in the circular $\mathcal{D}$ -region, defined in terms of its radius r , its distance γ from the imaginary axis, and its center $c = (-\sigma, 0)$, with $\sigma = \gamma + r$, as presented in Figure 1.

According to da Silva et al. (2012), if there exists a symmetric definite positive matrix P such that

$$\begin{bmatrix} A_{N\alpha}'P + PA_{N\alpha} + 2\gamma P & (A_{N\alpha} + \gamma I)'P \\ P(A_{N\alpha} + \gamma I) & -rP \end{bmatrix} < 0 \quad (5)$$

holds for $i = 1, \dots, N$, then $A_{N\alpha}$ is robust $\mathcal{D}$ -stable since all its system polytope eigenvalues are contained within the considered circular $\mathcal{D}$ -region.

Therefore, a strategy for dealing with the considered control problem is to design a robust SOF controller L such that (5) is satisfied, by defining $A_{N\alpha} = A_\alpha + B_\alpha LC_\alpha$.

Figure 1. The circular $\mathcal{D}$ -region.

Before properly discussing the proposed solution for the enunciated problem, we present the two following lemmas, which are necessary for the proof of our main results.

Lemma 1. (Finsler's Lemma). Consider $\mathcal{W} \in \mathbb{R}^n$, $\mathcal{S} \in \mathbb{R}^{n \times n}$, and $\mathcal{R} \in \mathbb{R}^{m \times n}$ with $\text{rank}(\mathcal{R}) < n$, where $\mathcal{R}^\perp$ is a basis for the null space of $\mathcal{R}$ (i.e. $\mathcal{R}\mathcal{R}^\perp = 0$).

Then, the following conditions are equivalent:

$$(i) \mathcal{W}'\mathcal{S}\mathcal{W} < 0, \quad \forall \mathcal{W} \neq 0, \quad \mathcal{R}\mathcal{W} = 0,$$

$$(ii) \mathcal{R}^\perp \mathcal{S} \mathcal{R}^\perp < 0,$$

$$(iii) \exists \eta \in \mathbb{R} : \mathcal{S} - \eta \mathcal{R}'\mathcal{R} < 0,$$

$$(iv) \exists \chi \in \mathbb{R}^{n \times m} : \mathcal{S} + \chi \mathcal{R} + \mathcal{R}'\chi' < 0,$$

where η and χ are additional variables (or multipliers).

Proof: See Skelton et al. (1997) and de Oliveira and Skelton (2001). ■

Lemma 2. If the following LMI

$$\Upsilon_{ii} < 0, \quad i = 1, 2, \dots, N, \quad (6)$$

and,

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \quad 1 \leq i < j \leq N \quad (7)$$

holds, then it is true that

$$\sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j \Upsilon_{ij} < 0. \quad (8)$$

Proof: See Tanaka et al. (1998). ■

3. MAIN RESULTS

In this section, we present the main contribution of this paper. It consists of an LMI-based strategy for deriving robust $\mathcal{D}$ -stabilising SOF controllers. The design procedure is based on the two-stage method (Aguilhari et al., 2012; Sereni et al., 2018). In this method, a robust state feedback gain K , which asymptotically stabilises

$$\dot{x}(t) = (A_\alpha + B_\alpha K)x(t) \quad (9)$$

must be obtained. Then, K is used as an input parameter for the second stage, in which the gain L is calculated.

The first stage gain K can be obtained by solving the set of LMI

$$\begin{aligned} W &= W' > 0, \\ A_i W + W A_i' + B_i Z + Z' B_i' + 2\beta W &< 0, \end{aligned} \quad (10)$$

for the variables $W \in \mathbb{R}^{n \times n}$ and $Z \in \mathbb{R}^{m \times n}$, considering a minimum decay rate $\beta > 0$. If (3) is feasible, then K is retrieved with $K = ZW^{-1}$ (Boyd et al., 1994).

Remark 1. Since in the two-stage method each design stage is performed separately, any other state feedback control design can be implemented in order to derive the first stage gain matrix. In fact, one might try to perform the first stage designing a parameter-dependent gain K_α , for example. Indeed, note that this first stage gain will not be implemented in practice, being used only as given data in the second stage. In this paper, just for simplicity we chose to consider a parameter-independent gain K .

Then, in the second stage, the SOF gain L may be obtained by using the gain K as an input parameter and solving the set of sufficient LMI conditions, proposed in Theorem 1, based on the results of da Silva et al. (2012), Leite et al. (2004), and Sereni et al. (2018).

Theorem 1. Assuming that there exists a state feedback gain K such that $A_\alpha + B_\alpha K$ is asymptotically stable, then there exists a static output feedback gain L such that $A_\alpha + B_\alpha LC_\alpha$ is robust $\mathcal{D}$ -stable, ensuring the closed-loop pole placement inside a circle with radius r and with center in $(\sigma, 0)$, where $\sigma = r + \gamma$, if there exist a symmetric matrix $P > 0$ and matrices F, G, H , and J such that

$$\begin{bmatrix} A'_i F' + F A_i + K' B'_i F' + F B_i K + \gamma P(2r + \gamma) & & \\ (r + \gamma)P - F' + G A_i + G B_i K & & \\ B'_i F' + J C_i - H K & & \\ P - G^* - G' & * & \\ B'_i G' & * & -H - H' \end{bmatrix} < 0 \quad (11)$$

for $i = 1, 2, \dots, N$.

In the affirmative case, the robust static output feedback gain is given by $L = H^{-1}J$.

Proof: Firstly, observe that supposing that (11) is feasible, then one can verify that H is invertible, since for every non-symmetric matrix M , if $M + M' < 0$, then M is invertible (Boyd et al., 1994).

Now, multiplying (11) by α_i , and summing every term in i , from $i = 1$ to $i = N$, and regarding that $\sum_{i=1}^N \alpha_j = 1$, we obtain

$$\begin{bmatrix} A'_\alpha F' + F A_\alpha + K' B'_\alpha F' + F B_\alpha K + \gamma P(2r + \gamma) & & \\ (r + \gamma)P - F' + G A_\alpha + G B_\alpha K & & \\ B'_\alpha F' + J C_\alpha - H K & & \\ P - G^* - G' & * & \\ B'_\alpha G' & * & -H - H' \end{bmatrix} < 0 \quad (12)$$

Pre and post multiplying (12) by T_α and T'_α , where T_α is defined as

$$T_\alpha = \begin{bmatrix} I & 0 & S'_\alpha \\ 0 & I & 0 \end{bmatrix}, \quad (13)$$

it follows that

$$\begin{bmatrix} \Psi_\alpha & \Phi_\alpha \\ * & P - G - G' \end{bmatrix} < 0, \quad (14)$$

where

$$\begin{aligned} \Psi_\alpha = & A'_\alpha F' + F A_\alpha + K' B'_\alpha F' + F B_\alpha K + \gamma P(2r + \gamma) \\ & + S'_\alpha B'_\alpha F' + S'_\alpha J C_\alpha - S'_\alpha H K + C'_\alpha J' S_\alpha \\ & - K' H' S_\alpha + F B_\alpha S_\alpha + S'_\alpha (-H - H') S_\alpha \end{aligned} \quad (15)$$

and,

$$\Phi_\alpha = (r + \gamma)P - F + A'_\alpha G' + K' B'_\alpha G' + S'_\alpha B'_\alpha G'. \quad (16)$$

Making $S_\alpha = H^{-1}J C_\alpha - K$, in (15) and (16), then we have

$$\begin{aligned} \Psi_\alpha = & \text{He}(A'_\alpha F' + K' B'_\alpha F' - K' H' (H^{-1}J C_\alpha - K)) \\ & + (C'_\alpha J' H^{-1} - K')(-H - H')(H^{-1}J C_\alpha - K) \\ & + \text{He}((C'_\alpha J' H^{-1} - K') B'_\alpha F') + \gamma P(2r + \gamma) \\ & + \text{He}((C'_\alpha J' H^{-1} - K') J C_\alpha) \end{aligned} \quad (17)$$

and,

$$\Phi_\alpha = (r + \gamma)P - F + (A_\alpha + B_\alpha H^{-1}J C_\alpha)' G'. \quad (18)$$

Expanding the products in (17), and considering the following equivalent relation $H^{-1}H = HH^{-1} = I = H^{-1}H' = H'H^{-1}$, comes

$$\begin{aligned} \Psi_\alpha = & (A_\alpha + B_\alpha H^{-1}J C_\alpha)' F' \\ & + F(A_\alpha + B_\alpha H^{-1}J C_\alpha) + \gamma P(2r + \gamma). \end{aligned} \quad (19)$$

Now, note that by making $L = H^{-1}J$ in (18) and (19), we can rewrite (14) as

$$\begin{bmatrix} \gamma P(2r + \gamma) + \text{He}(F(A_\alpha + B_\alpha LC_\alpha)) & * \\ (r + \gamma)P - F' + G(A_\alpha + B_\alpha LC_\alpha) & P - G - G' \end{bmatrix} < 0 \quad (20)$$

which, in turn, can be conveniently rearranged as

$$\begin{bmatrix} \gamma P(2r + \gamma) & (r + \gamma)P \\ (r + \gamma)P & 0 \end{bmatrix} + \text{He} \left(\begin{bmatrix} (A_\alpha + B_\alpha LC_\alpha)' \\ -I \end{bmatrix} \begin{bmatrix} F' & G' \end{bmatrix} \right) < 0. \quad (21)$$

By defining

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \chi, \quad \mathcal{S} = \begin{bmatrix} \gamma(2r + \gamma)P & (r + \gamma)P \\ (r + \gamma)P & P \end{bmatrix}, \quad (22)$$

$$\text{and, } \mathcal{R}_\alpha = \begin{bmatrix} A_\alpha + B_\alpha LC_\alpha & -I \end{bmatrix}, \quad (23)$$

we can rewrite (21) as $\exists \chi \in \mathbb{R}^{2n \times n}, \mathcal{S} + \chi \mathcal{R}_\alpha + \mathcal{R}'_\alpha \chi' < 0$, which corresponds to the condition (iv) of Finsler's Lemma, as stated on Lemma 1. Therefore, according to condition (i) of Finsler's Lemma, which is $\mathcal{W}' \mathcal{S} \mathcal{W} < 0, \forall \mathcal{W} \neq 0, \mathcal{R}_\alpha \mathcal{W} = 0$, and assuming that $\mathcal{W} = [x'(t) \dot{x}'(t)]'$, one may obtain

$$\begin{bmatrix} A_\alpha + B_\alpha LC_\alpha & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = 0 \quad \text{and} \quad (24)$$

$$[x'(t) \dot{x}'(t)] \begin{bmatrix} \gamma(2r + \gamma)P & (r + \gamma)P \\ (r + \gamma)P & P \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} < 0. \quad (25)$$

At first, observe that, according to (24) we have

$$\dot{x}(t) = [A_\alpha + B_\alpha LC_\alpha]x(t), \quad (26)$$

which corresponds to the considered system equation (4).

Moreover, note that (25) leads to

$$\begin{aligned} x'(t)\gamma P(2r + \gamma)x(t) + \dot{x}'(r + \gamma)Px(t) \\ + x'(t)(r + \gamma)P\dot{x} + \dot{x}'P\dot{x} < 0 \end{aligned} \quad (27)$$

$$\Leftrightarrow \gamma P(2r + \gamma) + (A_\alpha + B_\alpha LC_\alpha)'P(A_\alpha + B_\alpha LC_\alpha) + (r + \gamma)P(A_\alpha + B_\alpha LC_\alpha) + (A_\alpha + B_\alpha LC_\alpha)'(r + \gamma)P. \quad (28)$$

Now, defining $A_{\mathcal{N}\alpha} = A_\alpha + B_\alpha LC_\alpha$, from (28) we have that

$$\begin{aligned} \Leftrightarrow 2\gamma r P + \gamma^2 P + (A'_{\mathcal{N}\alpha} r + A'_{\mathcal{N}\alpha} \gamma)P \\ + (r P + \gamma P)A_{\mathcal{N}\alpha} + A'_{\mathcal{N}\alpha} P A_{\mathcal{N}\alpha} < 0, \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow 2\gamma P + \frac{\gamma^2}{r}P + A'_{N\alpha}P + \frac{\gamma}{r}A'_{N\alpha}P \\ &\quad + PA_{N\alpha} + \frac{\gamma}{r}PA_{N\alpha} + \frac{A'_{N\alpha}PA_{N\alpha}}{r} < 0, \\ &\Leftrightarrow A'_{N\alpha}P + PA_{N\alpha} + 2\gamma P \\ &\quad + \frac{1}{r}(\gamma A'_{N\alpha}P + \gamma^2 P + \gamma PA_{N\alpha} + A'_{N\alpha}PA_{N\alpha}) < 0, \\ &\Leftrightarrow A'_{N\alpha}P + PA_{N\alpha} + 2\gamma P \\ &\quad + \frac{1}{r}(A_{N\alpha} + \gamma I)'P(A_{N\alpha} + \gamma I) < 0, \\ &\Leftrightarrow \begin{bmatrix} A'_{N\alpha}P + PA_{N\alpha} + 2\gamma P & * \\ P(A_{N\alpha} + \gamma I) & -rP \end{bmatrix} < 0, \quad (29) \end{aligned}$$

which is a sufficient condition for the robust $\mathcal{D}$ -stabilisation (Leite et al., 2004) of SOF closed loop system (4). ■

In general, the SOF control problem has an intrinsic conservative mathematical formulation, being classified as a NP-hard problem. Therefore, it is natural to assume that finding a feasible solution in optimisation problems as solving LMI such as (11) is not a simple task.

Thus, the idea of finding less conservative conditions for dealing with SOF control is a very relevant issue. With this perspective, we propose relatively more relaxed LMI conditions for designing robust $\mathcal{D}$ -stabilising SOF controllers. The obtained results are based on considering a parameter-dependent Lyapunov function (P_α) and parameter-dependent additional Finsler's Lemma variables (F_α and G_α), as proposed in Theorem 2.

Theorem 2. Assuming that there exists a state feedback gain K such that $A_\alpha + B_\alpha K$ is asymptotically stable, then there exists a static output feedback gain L such that $A_\alpha + B_\alpha LC_\alpha$ is robust $\mathcal{D}$ -stable, ensuring the closed-loop pole placement inside a circle with radius r and with center in $(\sigma, 0)$, where $\sigma = r + \gamma$, if there exist symmetric matrices $P_i > 0$ and matrices F_i, G_i, H and J such that (30) and (31) are satisfied.

$$\begin{bmatrix} A'_i F'_i + F_i A_i + K' B'_i F'_i + F_i B_i K + \gamma P_i (2r + \gamma) \\ (r + \gamma) P_i - F'_i + G_i A_i + G_i B_i K \\ B'_i F'_i + J C_i - H K \\ P_i - G'_i - G'_i & * \\ B'_i G'_i & -H - H' \end{bmatrix} < 0 \quad (30)$$

for $i = 1, 2, \dots, N$.

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} \\ P_i - F'_i + G_i A_j + G_i B_j K + P_j - F'_j + G_j A_i + G_j B_i K \\ B'_i F'_j + J C_i + B'_j F'_i + J C_j - 2 H K \\ P_i + P_j - G'_i - G'_i - G'_j - G'_j & * \\ B'_i G'_j + B'_j G'_i & -2 H - 2 H' \end{bmatrix} < 0 \quad (31)$$

for $i = 1, 2, \dots, N-1, i = 1, 2, \dots, N-1$, and $j = i+1, i+2, \dots, N$, where $\Omega_{ij} + \Omega_{ji} = A'_i F'_j + F_i A_j + K' B'_i F'_j + F_i B_j K + \gamma P_i (2r + \gamma) + A'_j F'_i + F_j A_i + K' B'_j F'_i + F_j B_i K + \gamma P_j (2r + \gamma)$.

In the affirmative case, the robust static output feedback gain is given by $L = H^{-1}J$.

Proof: Note that (30) and (31) have the form $\Upsilon_{ii} < 0$ for $i = 1, 2, \dots, N$, and $\Upsilon_{ij} + \Upsilon_{ji} < 0$, for $1 \leq i < j \leq N$, respectively. Thus, according to Lemma 2, we have that

$$\begin{bmatrix} \sum_{i=1}^N \alpha_i(t) \sum_{j=1}^N \alpha_j(t) (\Delta_{ii}) \\ \sum_{i=1}^N \alpha_i(t) \sum_{j=1}^N \alpha_j(t) ((r+\gamma)P_i - F'_i + G_i A_i + G_i B_i K) \\ \sum_{i=1}^N \alpha_i(t) \sum_{j=1}^N \alpha_j(t) (B'_i F'_i + J C_i - H K) \\ \sum_{i=1}^N \alpha_i(t) \sum_{j=1}^N \alpha_j(t) (P_i - G'_i - G'_i) \\ \sum_{i=1}^N \alpha_i(t) \sum_{j=1}^N \alpha_j(t) (B'_i G'_i) \\ \sum_{i=1}^N \alpha_i(t) \sum_{j=1}^N \alpha_j(t) (-H - H') \end{bmatrix} < 0, \quad (32)$$

with $\Delta_{ii} = A'_i F'_i + F_i A_i + K' B'_i F'_i + F_i B_i K + \gamma P_i (2r + \gamma)$.

Developing the sums in (32), considering that $\sum_{i=1}^N \alpha_i(t) = \sum_{j=1}^N \alpha_j(t) = 1$, one can derive

$$\begin{bmatrix} A'_\alpha F'_\alpha + F_\alpha A_\alpha + K' B'_\alpha F'_\alpha + F_\alpha B_\alpha K + \gamma P_\alpha (2r + \gamma) \\ (r + \gamma) P_\alpha - F'_\alpha + G_\alpha A_\alpha + G_\alpha B_\alpha K \\ B'_\alpha F'_\alpha + J C_\alpha - H K \\ P_\alpha - G'_\alpha - G'_\alpha & * \\ B'_\alpha G'_\alpha & -H - H' \end{bmatrix} < 0 \quad (33)$$

Henceforward, the proof follows similarly as for Theorem 1. ■

4. EXAMPLES

In this section, we present some computational examples and the results of a practical implementation of the proposed strategy in illustrative and relevant control problems. In the following examples, the LMIs were programmed with MATLAB software, and solved via YALMIP interface (Lofberg, 2004), using the LMILab solver.

4.1 Example 1

In this first example, we consider the hypothetical uncertain LTI system presented in Leite et al. (2004). The system, described as in (3), can be represented as a convex combination of the vertices

$$A_1 = \begin{bmatrix} 2.439 & 0.683 \\ 0.933 & -2.787 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0.607 \\ 0.629 \end{bmatrix} \quad (34)$$

and,

$$A_2 = \begin{bmatrix} 0.102 & 1.223 \\ -9.614 & 2.027 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.370 \\ 0.575 \end{bmatrix}. \quad (35)$$

We also consider that only the state variable x_1 is available for measurement. Therefore, the output matrices for each vertex are defined as $C_1 = C_2 = [1 \ 0]$.

Intending to illustrate the reduction of conservativeness achieved with Theorem 2 LMIs, we present a feasibility analysis. This study consisted in attempting to solve the enunciated control problem by applying Theorem 1 and 2 in order to provide a robust SOF controller for a radius and decay rate specifications defined within the ranges $0 < r \leq 150$ and $0 \leq \beta = \gamma \leq 0.50$, respectively.

Figure 2. Feasibility region obtained in Example 1, using Theorem 1 ($\square$) and Theorem 2 ($\times$ and $\square$).

By analysing the experiment results presented in Figure 2, we can clearly conclude that Theorem 2 could provide a better performance than Theorem 1 in terms of feasibility, showing that it indeed consists in less conservative controller synthesis conditions. This conclusion is based on the fact that by using Theorem 2 a solution was obtained for a larger diversity of, and also more restricted, $\mathcal{D}$ -region specifications (defined by the pair of scalars r and $\gamma = \beta$) in the controller design.

4.2 Example 2

In this new experiment, we considered the same hypothetical system defined in Example 1. However, we set a particular design specification in order to illustrate the technique efficacy in providing a robust pole placement for the considered uncertain LTI system. The desired circular $\mathcal{D}$ -region for robust pole placement is defined with radius $r = 30$ and $\gamma = 0.30$. Applying the proposed strategy, the first stage gain K derived solving the LMI conditions in (3), with $\beta = 0.30$, was $K = [-72.2644 \quad 30.8434]$. Then, using K as an input for the second stage, and solving the LMIs (30) and (31) in Theorem 2 with the specified values of r and γ , the robust SOF gain synthesised was $L = -72.6990$. Note that since only one state variable is being considered for feedback, we have a simple scalar controller L .

The eigenvalues (λ) of the closed-loop system with the designed robust SOF gain were allocated as presented in the eigenvalues cloud in Figure 3, obtained for all convex combinations of the closed-loop system vertices, according to the unitary simplex defined in (2). As one can observe, all eigenvalues are placed inside the specified region (see the right chart in Figure 3 for detail), proving the method efficacy.

4.3 Example 3

This final example presents a practical implementation of the proposed method. The objective was to design a robust SOF controller for a QUANSER[®] Active Suspension.

This system represents a 1/4 model of a vehicle, as showed in Figure 4. In the presented schematic diagram, M_s is the sprung mass, representing 1/4 of vehicle body (load) mass, M_{us} is the unsprung mass that represents the tire of the

Figure 3. Closed-loop eigenvalues of the uncertain system $A_\alpha + B_\alpha LC_\alpha$ ($\times$), with system vertices eigenvalues highlighted ($\diamond$) (left chart), and eigenvalues cloud close up (right chart).

quarter-car model, and k_s , b_s , k_{us} , and b_{us} , are the springs and dampers in the model assembly. $z_{us}(t)$ and $z_s(t)$ are the unsprung and sprung mass positions related to each shown reference level (Quanser, 2009).

The objective is to mitigate the effects caused by the road surface ($z(t)$) on the car's suspension travel and road handling. In the considered model, these performance parameters are related to the relative movement between the vehicle body and tire ($z_s - z_{us}$), and to the displacement between the tire and the road surface ($z_{us} - z_r$), respectively. This goal may be achieved by properly applying a control command to drive a DC motor that acts as the active element in the suspension system, which in turn, exerts a force $F_c(t)$ to control the plates displacements.

Figure 4. Schematic diagram of the QUANSER[®] Active Suspension System.

In (36) we have a state space model for the active suspension system considered in this example.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-k_s}{M_s(\alpha)} & \frac{-b_s}{M_s(\alpha)} & 0 & \frac{b_s}{M_s(\alpha)} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{M_{us}} & \frac{b_s}{M_{us}} & \frac{-k_{us}}{M_{us}} & \frac{-(b_s+b_{us})}{M_{us}} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{\rho(\alpha)}{M_s(\alpha)} \\ 0 \\ \frac{-\rho(\alpha)}{M_{us}} \end{bmatrix} u(t),$$

$$y(t) = [0 \ 1 \ 0 \ 0] x(t). \quad (36)$$

We assume that the sprung mass is an uncertain parameter lying within the interval $1.455 \text{ kg} \leq M_s(\alpha) \leq 2.45 \text{ kg}$. Moreover, we take into account that the actuator, which applies the control signal $u(t)$, is susceptible to present

a fault in power delivery to the DC motor, implying in the attenuation of the control signal. In this example, we assume that a maximum attenuation of 30 % in the control signal is possible to occur.

This scenario can be mathematically modelled by considering a second uncertain parameter $\rho(\alpha)$, such that $0.7 \leq \rho(\alpha) \leq 1$, which affects all entries of the input matrix (as seen in (36)), impressing the control signal attenuation due to the power failure on the system dynamics. Furthermore, it is assumed that only one of the four system state variables is available for measurement (i.e. x_2).

With two uncertain parameters, the active suspension system may be represented as a convex combination of four vertices, defined accordingly to the matrices

$$A_1 = A_3 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.56 & -5.1546 & 0 & 5.1546 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix},$$

$$A_2 = A_4 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.35 & -3.0612 & 0 & 3.0612 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0.6873 \\ 0 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.4082 \\ 0 \\ -1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 \\ 0.4811 \\ 0 \\ -0.7 \end{bmatrix}, \quad \text{and} \quad B_4 = \begin{bmatrix} 0 \\ 0.2857 \\ 0 \\ -0.7 \end{bmatrix},$$

considering the parameters values specified in Table 1. (Quanser, 2009).

Table 1. QUANSER® Active Suspension parameters.

Parameter	Value	Parameter	Value
M_s	2.45 kg	k_{us}	2500 N/m
M_s	2.45 kg	b_s	7.5 Ns/m
M_{us}	1.0 kg	b_{us}	5.0 Ns/m
k_s	900 N/m		

By applying the proposed method, a first stage gain matrix $K = [-34.0143 \ -30.2183 \ -194.7134 \ -5.0741]$ can be obtained using $\beta = 0.45$ in (3). And, solving Theorem 2 LMIs (30) and (31) with $r = 310$ and $\gamma = 0.45$, a robust SOF gain $L = -81.6010$ is derived. Again, observe that L consists in a simple scalar value, as we are considering only one state variable for composing the feedback loop.

In fact, the proposed strategy was able to perform the robust pole placement, as all system polytope eigenvalues were fitted in the defined $\mathcal{D}$ -region, as showed in Figure 5.

Figure 5. Closed-loop eigenvalues of active suspension system polytope (x), system vertices eigenvalues highlighted (◇), and eigenvalues cloud close up (inset).

Regarding the practical implementation of the designed robust SOF controller, we present the dynamic performance of the closed-loop active suspension system. For the considered experiment, the road profile ($z_r(t)$) was set to a square wave with 0.02 m of amplitude and frequency of 0.25 Hz. Additionally, a fault of 30% power loss was programmed to occur at $t = 16$ s. Figure 6 shows the system behaviour with $M_s(\alpha)$ set to its minimum value, and Figure 7 brings the results obtained with $M_s(\alpha)$ adjusted to its maximum value.

One can observe that in both situations the closed-loop response is stable and has a better performance when compared to the open-loop case, as the oscillations in both blue and red plates were significantly reduced. Moreover, even in the occurrence of actuator fault (30% in power loss), the closed-loop system presents a stable and better transient when compared to the open-loop scenario. The effect of the fault occurrence can be seen in the lower peak values of the effective force provided by the DC motor, as showed in the second chart in Figures 6 and 7.

Figure 6. Active suspension dynamic response with $M_s(\alpha) = 1.455$ kg in: open-loop (0-8s); closed-loop (8-16s); fault: 30% power loss in the actuator (16-24s).

Figure 7. Active suspension dynamic response with $M_s(\alpha) = 2.45$ kg in: open-loop (0-8s); closed-loop (8-16s); fault: 30 % power loss in the actuator (16-24s).

5. CONCLUSION

A new robust $\mathcal{D}$ -stabilising SOF control design LMI-based method was proposed in this paper. Differently from other papers that also addressed the robust pole placement in a circular $\mathcal{D}$ -region for uncertain LTI systems, our LMI conditions proposed in Theorem 1 offers a solution for incomplete state measurement via SOF control. In fact, any linear combination of the available states can be used for feedback, because in our method the matrix C is not necessarily composed by identities and zeros, but can assume any structure, including a polytopic one. Conservativeness reduction could be achieved by considering both PDLF and PDFV approaches. This fact was verified since applying Theorem 2 LMIs provided a wider feasibility region than with Theorem 1, for the presented example. Moreover, the practical applicability of our strategy was proven with the successful controller design and implementation for an active suspension system. The derived robust SOF controller was able to robustly allocate all system polytope eigenvalues and mitigate the oscillations of both suspension levels, even considering an uncertain load mass, the occurrence of actuator failure, and feedback loop with only one of four system state variables. For future works, less conservative first stage gain design may be employed to improve feasibility in the second stage.

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