

NOVEL PROCEDURES FOR GRAPH EDGE-COLOURING

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ABSTRACT

We present novel polynomial-time procedures for graph edge-colouring, a combinatorial optimisation problem with many applications in industry which has raised challenging conjectures. We show that all graphs whose vertices have local degree sum not *too* large can be optimally edge-coloured in polynomial time (the *local degree sum* of a vertex is the sum of the degrees of its neighbours). We also show that the set of the graphs satisfying this condition includes *almost every* graph (under the uniform distribution). We present further results on edge-colouring join graphs, chordal graphs, circular-arc graphs, and complementary prisms, whose proofs yield polynomial-time algorithms. Our results contribute towards settling the *Overfull Conjecture*, a main open problem on edge-colouring simple graphs (also known as *1-Factorisation Conjecture* in the context of d -regular graphs with $d \geq n/2$). Finally, we also present some results on total colouring, proving upper bounds which can be achieved in polynomial time.

KEYWORDS. Combinatorial optimization (MSC 90C27), Colouring of graphs and hypergraphs (MSC 05C15), Graph algorithms (MSC 05C85), Graph operations (MSC 05C76).

RESUMO

Apresentamos novos procedimentos polinomiais para coloração de arestas de grafos, um problema de otimização combinatória com muitas aplicações na indústria que levantou conjecturas desafiadoras. Mostramos que todo grafo cujos vértices possuem soma local de graus não alta *de-mais* pode ter suas arestas otimamente coloridas em tempo polinomial (a *soma local de graus* dum vértice é a soma dos graus dos seus vizinhos). Mostramos que o conjunto dos grafos que satisfazem esta condição inclui *quase todo* grafo (sob distribuição uniforme). Apresentamos também resultados nos grafos-junção, cordais, arco-circulares, e prismas complementares, cujas provas levam a algoritmos polinomiais. Nossos resultados contribuem na direção de provar a *Conjectura Overfull*, um principal problema em aberto em coloração de arestas de grafos simples (também conhecido

por *Conjectura da 1-Fatoração* no contexto de grafos d -regulares com $d \geq n/2$). Finalmente, apresentamos alguns resultados em coloração total, provando limitantes superiores que podem ser atingidos em tempo polinomial.

PALAVRAS CHAVE. Otimização combinatória (MSC 90C27), Coloração de grafos e hipergrafos (MSC 05C15), Algoritmos em grafos (MSC 05C85), Operações em grafos (MSC 05C76).

Doctoral thesis advised by Renato Carmo (renato@inf.ufpr.br) and co-advised by André L. P. Guedes (andre@inf.ufpr.br), and defended in 5 December 2018, awarding the author the degree of Doctor in Computer Science conferred by the Graduate Programme in Informatics, Department of Informatics, Faculty of Exact Sciences, Federal University of Paraná.

1. Introduction

Computing an optimal edge-colouring of a graph is an NP-hard [Holyer, 1981] combinatorial optimisation problem with applications in network protocols [Erlebach e Jansen, 2001; Gandham et al., 2005] and task scheduling in industry [Williamson et al., 1997]. There has been much work aimed at identifying sets of instances for which the problem becomes polynomial (e.g. [Chetwynd e Hilton, 1989; Bodlaender, 1990; Ortiz Z. et al., 1998]) and sets of instances for which NP-hardness remains (e.g. [Cai e Ellis, 1991; Koreas, 1997; Machado et al., 2010]). We refer the reader to the doctoral thesis, Sect. 1.6, p. 26, for an extensive table on the complexity of edge-colouring restricted to several graph classes.

By Vizing's Theorem [Vizing, 1964], the chromatic index of any simple graph G (i.e. the least k for which G is k -edge-colourable) is either its maximum degree Δ or $\Delta + 1$. The graph G is said to be *Class 1* in the former case, or *Class 2* in the latter. The *core* (i.e. the subgraph induced by the vertices of maximum degree) of *almost every*¹ graph is unitary [Erdős e Wilson, 1977], hence acyclic, and graphs with acyclic core are *Class 1* and admit a polynomial-time exact edge-colouring algorithm [Fournier, 1977], implying that the set of instances for which edge-colouring is hard includes *almost no* graph.

Intriguingly, edge-colouring is NP-hard even for graph classes such as perfect graphs [Cai e Ellis, 1991]. For many other classes, even well-structured ones such as cographs or chordal graphs, the complexity of the problem remains open, with apparently few progress on long-standing conjectures despite much work (see Chap. 2 of the thesis).

A main open problem on edge-colouring simple graphs (also known as *1-Factorisation Conjecture* in the context of d -regular graphs with $d \geq n/2$). If this conjecture holds, it implies that edge-colouring is polynomially solvable for graphs with $\Delta > n/3$. The results which we have achieved contribute towards settling this conjecture and their constructive proofs yield polynomial-time edge-colouring algorithms for *almost every* graph with cycles in the core (recall that edge-colouring is already solved for graphs with acyclic core [Fournier, 1977]).

This paper is organised as follows: in the remaining of this section we present some definitions which appear in the sequel; then, in Section 2 we present our main results and discuss their impact; in Section 3 we briefly present our recolouring procedure, which is the main technique used in the proofs for our main results; at last, we conclude in Section 4 with final remarks.

Definitions

All graphs considered in this text are simple. The set of neighbours of a vertex u in a graph G is denoted $N_G(u)$. A *major* of G is a vertex of maximum degree in G , and the *local degree sum* of a vertex is the sum of the degrees of its neighbours. The *core* of G is denoted $\Lambda[G]$. Other graph-theoretical concepts follow their usual definitions (see Sect. 1.3 of the thesis).

An n -vertex graph with maximum degree Δ is said to be *overfull* if it has more than $\Delta \lfloor n/2 \rfloor$ edges. A graph G is said to be *subgraph-overfull* (shortly, *SO*) if it has an *overfull* subgraph H with $\Delta(H) = \Delta(G)$. Deciding if a graph is *SO* can be done in polynomial time [Padberg e Rao, 1982]. Clearly, every *SO* graph G is *Class 2*, but examples of non-*SO* *Class 2* graphs are known (e.g. the Petersen graph). The *Overfull Conjecture* states that being *SO* is equivalent to being *Class 2* for n -vertex graphs with $\Delta > n/3$ [Chetwynd e Hilton, 1984, 1986; Hilton e Johnson, 1987].

2. Main results and impact

In this section we present some of the results achieved and briefly discuss their impact.

¹When we say that *almost every* graph satisfies a property $\mathcal{P}$, we mean that the probability of a uniformly sampled $(\mathcal{G}_{n,1/2})$ n -vertex graph to satisfy $\mathcal{P}$ goes to 1 as n goes to ∞ .

Theorem 1 (Theorem 1.10 of the thesis). *Let $\mathcal{X}$ be the class of the graphs with maximum degree Δ whose majors have local degree sum bounded above by $\Delta^2 - \Delta$. All graphs in $\mathcal{X}$ are Class 1.*

We also show that *almost every* graph G is in $\mathcal{X}$ even conditioning on G having cycles in the core, that is, even when restricted to the instances which have not been ruled out by the aforementioned results of [Erdős e Wilson, 1977] and [Fournier, 1977] (see the proof for Theorem 3.9 in the thesis).

The proof of Theorem 1 lies on a novel recolouring procedure with which an optimal edge-colouring can be constructed in polynomial time edge by edge for any graph in $\mathcal{X}$ (see Sect. 3 of this text and Chap. 3 of the thesis). Further, the development of this recolouring procedure contributes towards settling the Overfull Conjecture. As discussed in Sect. 6.1 of the thesis, [Niessen, 1994] suggests that a step towards a proof for the Overfull Conjecture is to prove the following.

Conjecture 1. *If u is a vertex of a graph G adjacent to at most one major of G with local degree sum at least $\Delta^2 - \Delta + 2$, and if $\chi'(G - u) \leq \Delta$, then G is Class 1.*

Using our recolouring procedure we have proved a slightly weaker form of this conjecture, just replacing the lower bound $\Delta^2 - \Delta + 2$ with $\Delta^2 - \Delta + 1$ (see the proof for Theorem 3.17 in the thesis). This and the fact that almost every graph is in $\mathcal{X}$ enhance the importance of our recolouring procedure.

Below we state our main result on complementary prisms.

Theorem 2 (Theorem 1.13 of the thesis). *A complementary prism can be Class 2 only if it is a regular graph distinct from the K_2 .*

We have also proved that no complementary prism can be *SO* (Theorem 4.30 of the thesis). This implies that a *Class 2* (regular) complementary prism would serve as an example of our generalisation of the definition of the snarks for d -regular graphs for $d \geq 3$ odd (see Sect. 4.4 of the thesis). Recall that snarks constitute an important graph class much related to the history of the *Four Colour Theorem* (see Chap. 2 of the thesis).

The result below is a special case of a conjecture on edge-colouring chordal graphs, open for more than two decades, which states that all chordal graphs of odd maximum degree are *Class 1* [Figueiredo et al., 1995].

Theorem 3 (Theorem 4.36 of the thesis). *Except for the K_3 , all chordal graphs with maximum degree $\Delta \leq 3$ are Class 1.*

Theorem 6 (i) and (ii) below contribute towards settling the Overfull Conjecture for join graphs and cographs (see Chap. 2 and 4 of the thesis), since in these theorems we prove that some graphs which we know that are not *SO* are in fact *Class 1*.

Theorem 4. *Let $G := G_1 * G_2$ be a join graph with $n_1 \leq n_2$ and $\Delta_1 \geq \Delta_2$ (we define $n_i := |V(G_i)|$ and $\Delta_i := \Delta(G_i)$ for $i \in \{1, 2\}$). The following are sufficient conditions for G to be Class 1:*

- (i) (Theorem 4.9 of the thesis) $\Lambda[G_1]$ is acyclic and, being $T_1, \dots, T_k$ the connected components of $\Lambda[G_1]$,

$$|\{u \in V(\Lambda[G_1]) : d_{\Lambda[G_1]}(u) > 1\}| + |\{T_i : |V(T_i)| = 2\}| \leq n_2 - |V(\Lambda[G_2])|;$$

- (ii) (Theorem 4.18 of the thesis) $n_2 - n_1 \geq 2$.

We remark that the proofs for all the aforementioned results are constructive and yield polynomial-time edge-colouring algorithms. We also present some results on total colouring (Theorems 5 and 6 below), settling upper bounds for this combinatorial optimisation problem when restricted to join graphs and cobipartite graphs (see Figure 1) and proving that colourings respecting these bounds can be achieved in polynomial time.

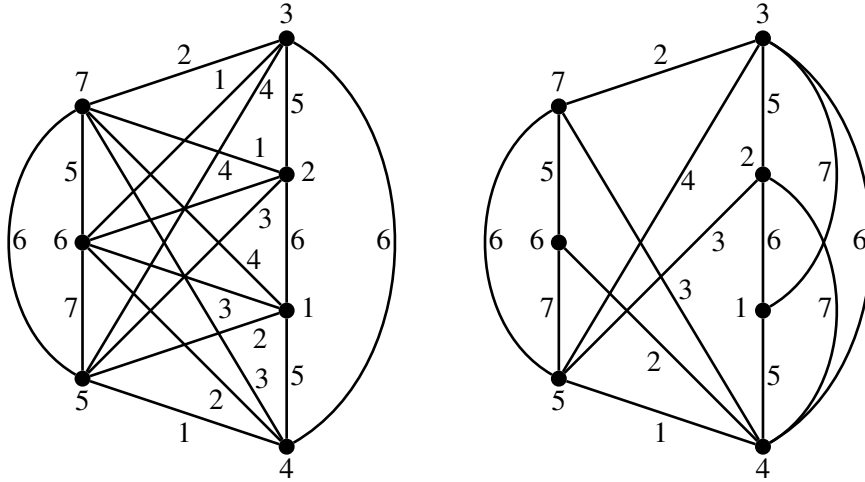


Figure 1: Optimal total colourings for the join graph $K_3 * C_4$ and for a cobipartite graph with $n_1 = 3$ and $n_2 = 4$

Theorem 5 (Theorem 5.2 of the thesis). *Let G be a connected cobipartite graph with $V(G) = V_1 \cup V_2$, wherein V_1 and V_2 are two disjoint cliques with $|V_1| =: n_1$ and $|V_2| =: n_2$. Let also $B_G := G[\partial_G(V_1)] = G[\partial_G(V_2)]$, a (not necessarily complete) bipartite graph, and let $\Delta_i^B := \max_{u \in V_i} d_{B_G}(u)$ for $i \in \{1, 2\}$. Then, $\chi''(G) \leq \max\{n_1, n_2\} + 2(\max\{\Delta_1^B, \Delta_2^B\} + 1)$.*

Theorem 6 (Theorem 5.4 of the thesis). *Let G be the join of two disjoint graphs G_1 and G_2 with, respectively, n_1 and n_2 vertices and maximum degrees Δ_1 and Δ_2 , but without assuming this time that $n_1 \leq n_2$. Let B_G be the complete bipartite graph $G - (E_1 \cup E_2)$. Let also*

$$P(G_1, G_2) := \min\{\Delta_1 + \Delta_2 + 1, \max\{\chi'_1, \chi''_2\}\}. \quad (1)$$

Then, $\chi''(G) \leq \max\{n_1, n_2\} + 1 + P(G_1, G_2)$.

The results above on total colouring use novel decomposition techniques and results on other graph colouring problems, mainly list-colouring problems, in order to colour each part of the decomposed graph respecting the upper bounds. In the thesis we also discuss how the upper bounds presented for these graph classes are better than the upper bounds for the total chromatic number of general graphs.

3. The recolouring procedure

We present a sketch of our recolouring procedure (see Chap. 3 of the thesis for the details). Throughout this section, let $\varphi: E(G) \setminus \{uv\} \rightarrow \mathcal{C}$ be a $\Delta(G)$ -edge-colouring of $G - uv$ for some edge $uv \in E(G)$, which is the edge about to be coloured by the procedure.

If the vertex u and all its neighbours miss at least one colour of $\mathcal{C}$ each (we say that a vertex x *miss* a colour α if no edge incident to x is coloured α), then we can use Vizing's recolouring

procedure in order to obtain a colour of $\mathcal{C}$ to assign to uv without creating colour conflicts [Vizing, 1964]. However, since $|\mathcal{C}| = \Delta(G)$, the neighbours of u in $G - uv$ considered by the procedure may not *actually* miss at least one colour of $\mathcal{C}$, but can still be handled as long as they *virtually* miss at least one colour of $\mathcal{C}$, as defined below. Our procedure extends Vizing's in the sense that if every considered vertex actually misses a colour, then our procedure behaves exactly as Vizing's.

Definition 1. A sequence $v_0, \dots, v_k$ of *distinct* neighbours of u in G is a *recolouring fan* for uv if $v_0 = v$ and, for all $i \in \{0, \dots, k-1\}$: either v_i actually misses the colour $\alpha_i := \varphi(uv_{i+1})$; or v_i misses the colour $\alpha_i := \varphi(uv_{i+1})$ *virtually*, that is, $i > 0$ and $\varphi(v_i w_i) = \alpha_i$ for some $w_i \in N_G(v_i) \setminus \{v_{i-1}\}$ which actually misses α_{i-1} .

We show that if there is a recolouring fan $v_0, \dots, v_k$ for uv such that v_k misses (actually or virtually) a colour which is

(Condition 1) either missing at u ,

(Condition 2) or missing (actually or virtually) at v_j for some $j < k$,

then the edges of $G - uv$ can be recoloured in order to obtain a colour to be assigned to uv without creating colour conflicts. Although the proof under Condition 1 goes quite straightforwardly, as Fig. 2 illustrates, several challenging complications rise in order to guarantee the proof under Condition 2, which we have successfully managed (see Lemmas 3.1–3.7, p. 51–59 of the thesis). Fig. 3

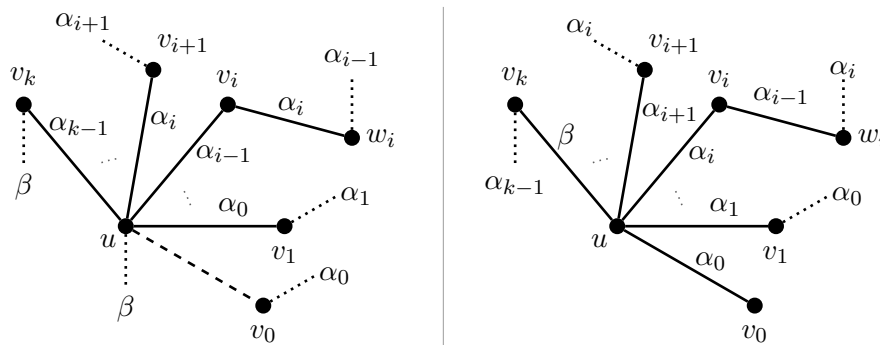


Figure 2: A recolouring fan $v_0, \dots, v_k$ for uv such that v_k actually misses a colour β which is missing at u . The dotted lines indicate the colours actually missing at the vertices, and the dashed line the edge to be coloured.

illustrates one of these complications.

We show that if all neighbours of u in G have local degree sum bounded above by $\Delta^2 - \Delta$, then we can always recolour the edges of $G - uv$ in order to construct a recolouring fan for uv satisfying Condition 1 or Condition 2 (see Lemma 3.8, p. 59–60, of the thesis). This yields the constructive proof of Theorem 1.

The proof that almost every graph is in $\mathcal{X}$ follows from the fact that almost every graph G has a single vertex u of degree at least $\frac{1}{2}((n-1) + (1-\varepsilon)n \ln n)^{\frac{1}{2}}$ for any $\varepsilon > 0$ [Erdős e Wilson, 1977], but no vertex of degree exactly $d_G(u) - 1$, as we show (see Theorem 3.9 of the thesis for details).

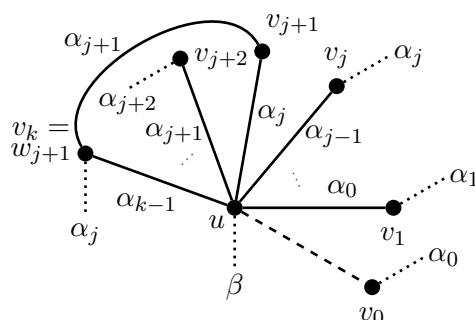


Figure 3: A recolouring fan $v_0, \dots, v_k$ for uv such that v_k misses α_j for some $j < k$, but the vertex v_{j+1} misses α_{j+1} virtually and $v_k = w_{j+1}$.

4. Conclusion

We have presented a novel recolouring procedure for edge-colouring, with which we have achieved the result stated in Theorem 1 and a slightly weaker form of Conjecture 1, contributing towards settling the Overfull Conjecture. We have also shown that almost every graph is covered by Theorem 1 and thus can be edge-coloured in polynomial time.

We have also achieved results on edge-colouring graph classes such as complementary prisms, chordal graphs, and join graphs, contributing towards settling the Overfull Conjecture for those classes (see Chap. 4 of the thesis for more results achieved).

Finally, we suspect the following stronger form of Theorem 6(i) (Theorem 4.9 of the thesis), for which we also develop a recolouring procedure (see Sect. 4.3 of the thesis).

Conjecture 2 (Conjecture 1.11 of the thesis). *Let $G := G_1 * G_2$ be a join graph with $n_1 \leq n_2$ and $\Delta_1 \geq \Delta_2$. If $\Lambda[G_1]$ is acyclic, then G is Class 1.*

Acknowledgements

This work was partially supported by UFFS (grant 23205.001243/2016-30), CNPq (grant 428941/2016-8), UFPR/PPGInf, and PESC/CAPES (grant 0588/2018).

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This doctoral work has also been classified to the Finals of the 32th *Thesis and Dissertation Contest* of Brazilian Computer Science (CTD/SBC).

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