

ON EDGE AND TOTAL COLORING OF GRAPHS*

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RESUMO

Uma coloração de arestas (coloração total, respectivamente) é a atribuição de cores às arestas (vértices e arestas) de um grafo, tal que arestas adjacentes (elementos incidentes e adjacentes) não recebam a mesma cor. Se a diferença entre a cardinalidade de quaisquer duas classes de cor for no máximo um em uma coloração total, então ela é dita equilibrada. O menor inteiro positivo que satisfaz essa condição é dito o número cromático total equilibrado do grafo. Para tal valor, Wang (2002) conjecturou um limite superior. Da Silva (2018) fez uma revisão de literatura sobre os principais resultados conhecidos sobre coloração de arestas e determinou o número cromático total equilibrado de todos os grafos r -partidos completos p -balanceados. Neste trabalho nós apresentamos uma versão compacta do trabalho de da Silva com foco nos novos resultados.

PALAVRAS CHAVE. Grafos r -partidos completos p -balanceados. Número cromático total equilibrado. Algoritmo.

Tópicos. Coloração de grafos. Coloração Total. Coloração Total Equilibrada

ABSTRACT

An edge coloring (total coloring, respectively) is the assignment of colors to the edges (vertices and edges) of a graph, so that adjacent edges (incident and adjacent elements) do not receive the same color. If the difference between the cardinality of any two color classes is at most one in a total coloring, then it is said to be equitable. The smallest positive integer that satisfies this condition is said to be the graph's equitable total chromatic number. For such value, Wang (2002) conjectured an upper bound. Da Silva (2018) did a literature review of the main known results of edge coloring and determined the equitable total chromatic number of all complete r -partite p -balanced graphs. In this work we present a compact version of da Silva's work with focus on the new results.

KEYWORDS. Complete r -partite p -balanced graphs. Equitable total chromatic number. Algorithm.

Paper topics. Graph coloring, Total Coloring, Equitable Total Coloring.

*This work is an extended abstract based on the author's Master's thesis entitled "Um estudo sobre coloração de arestas e coloração total de grafos" ("A study on edge and total coloring of graphs"). The work was supervised by Dr. Simon Richard Griffiths (Pontifical Catholic University of Rio de Janeiro - PUC-Rio) and Dr. Diana Sasaki Nobrega (Rio de Janeiro State University - UERJ) and defended on November 30th, 2018 as partial requirement to obtain the Master's degree from the Graduate Program in Mathematics of the Department of Mathematics of PUC-Rio.

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1. Introduction

In graph theory, graph coloring problems are about assigning colors to the elements of the graph satisfying some properties. This field of graph theory was initially motivated by the perception that four colors suffice to color any map that can be drawn on the plane so that regions that share a border are colored with distinct colors. Such perception generated a conjecture whose veracity was proved only over one century after it was proposed. Such problem is known as the Four Color Problem. Before the result was finally proved, several erroneous proofs were proposed.

It can be said that the Four Color Problem gave rise to a sub-area of graph theory called graph coloring, in which it is desired to assign colors to elements of the graphs subject to certain constraints. In general, the goal is that conflicting elements do not receive the same color. In the case of coloring a map, for example, the conflict is represented by the border between two regions. In terms of graphs, adjacent elements or incidents must be assigned to different colors.

In [da Silva, 2018] there is a literature review of edge coloring and new results on total coloring are presented. The portion of the Master's thesis that contains new results is a joint work with Dr. Simone Dantas (Universidade Federal Fluminense - UFF) and Dr. Diana Sasaki (Universidade do Estado do Rio de Janeiro - UERJ). We define a *k-edge coloring* as the assignment of colors to the edges of a graph so that adjacent edges receive different colors. The least positive integer k for which a graph G admits a k -edge coloring is said to be the *chromatic index* of G and is denoted by $\chi'(G)$. The chromatic index of a simple graph can be either $\Delta(G)$ or $\Delta(G) + 1$, which allows us to classify simple graphs in two categories, which are *class 1* and *class 2* graphs, respectively. Another type of graph coloring is the *k-total coloring* of a graph G , that consists of the assignment of k colors to the vertices and edges of G so that adjacent and incident elements receive different colors. Analogously to the previous coloring we defined, the *total chromatic number* of a graph G is the smallest positive integer k for which G admits a k -total coloring and it is denoted by χ'' .

2. On edge coloring

A particular kind of edge coloring is the *list edge coloring*. Suppose that for each edge e of a graph G is given a list $L(e)$ of colors allowed in e . If there exists an edge coloring of G so that the color given to each edge e is an element of $L(e)$, then we say that the coloring is a coloring of G by the lists $(L(e))_{e \in E}$. The smallest positive integer k for which G has an edge coloring by any family of lists $(L(e))_{e \in E}$ (with $|L(e)| = k \forall e \in E$) is called the *list chromatic index* of G and is denoted by $\chi'(G)$, according to the notation found in [Galvin, 1995]. According to Bondy and Murty [Bondy e Murty, 2008], several authors, including Vizing, Gupta, Albertson and Collins, established the List Coloring Conjecture (LCC) regarding the list chromatic index of graphs with no loops. In the second chapter of [da Silva, 2018], the author exhibits the main results of edge coloring and list edge coloring, focusing, particularly, on the known facts that are more relevant to determine the list chromatic index of bipartite multigraphs. In [Galvin, 1995], the author verified the LCC for that class of graphs and da Silva presents a study on that result.

3. On equitable total coloring of $K_{r \times p}$

It has been conjectured in [Behzad, 1965] and [Vizing, 1968] that every simple graph G satisfies $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$, where χ'' is called the total chromatic number of G and denotes the least integer for which G admits a total coloring. This conjecture is known as the Total Coloring Conjecture (TCC). If we add the constraint that the difference between the cardinalities of any two color classes is either 0 or 1 in a total coloring, then it is said to be *equitable*. Similarly to the TCC, The Equitable Total Coloring Conjecture (ETCC), due to [Wang, 2002] states that every simple graph satisfies $\Delta + 1 \leq \chi_e''(G) \leq \Delta + 2$, where χ_e'' is called the equitable total chromatic number of a graph and is the smallest integer for which a graph admits an equitable total coloring. Several

papers have addressed equitable total coloring of graphs. This motivated our investigation on the equitable total coloring of $K_{r \times p}$.

Before we show that $K_{r \times p}$ has $\chi''_e = \Delta + 1$ for r and p even ($r \geq 4$), we present some preliminary information that will be useful. We refer the interested reader to [da Silva et al., 2018] for more details on notations and definitions. When a color is given to a vertex or to an edge that incides in that vertex, we say that the color was *represented* in such vertex. In every $(\Delta + 1)$ -equitable total coloring of regular graphs, all the colors must be represented in every vertex, since we must use Δ colors for the incident edges to a given vertex and another color to the vertex itself.

We denote the j -th vertex of the i -th independent set of V by x_{ij} . Thus, we have $X_i = \{x_{i1}, x_{i2}, \dots, x_{ip}\}$. There is not a unique way to draw a graph. Despite this, always that we draw a graph $K_{r \times p}$, we will display the vertices as in a matrix with r columns and p rows, in which each column corresponds to a part X_i and vertex x_{ij} will be on the j -th row and on the i -th column. In Figure 1 we give an example to help the reader to understand what we mean.

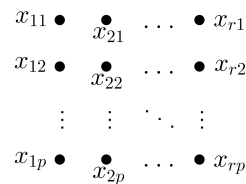


Figure 1: Vertex display for $K_{r \times p}$

Thinking on the vertices display as in a matrix, we define a *horizontal edge* as an edge of type $x_{ab}x_{cb}$. Also, if we mention vertices of a given row (column b) a , we will be referring to $x_{1a}, x_{2a}, \dots, x_{ra}$, with $1 \leq a \leq p$ ($x_{b1}, x_{b2}, \dots, x_{bp}$, with $1 \leq b \leq r$). The vertices display that we chose is helpful not only for making it easier to visualize the graphs, but also because it assists in the comprehension of the following definition.

A horizontal matching of distance i linking vertices of rows a and b ($1 \leq a < b \leq p$) is the matching $\{x_{ja}x_{j+i,b} | 1 \leq j \leq r\}$, where index $j + i$ is taken modulo r . In Figure 2 we exhibit horizontal matchings of distance 1 and 2 linking vertices of rows 1 and 2 of $K_{3 \times 2}$.

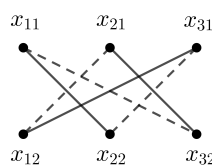


Figure 2: Horizontal matchings of distance 1 and 2 linking vertices of rows 1 and 2 in $K_{3 \times 2}$ represented, respectively, by continuous and dashed lines

We observe that $1 \leq i \leq r - 1$. Throughout the text, when we mention a result taken modulo α , if the result is congruent to 0 modulo α , then we will use α instead of 0. Since $1 \leq j \leq r$, each horizontal matching has r edges. Therefore, if we apply a color c on a horizontal matching of distance linking vertices of any two rows, then c is represented in every vertex of the given rows. Note that given two rows, there are $r - 1$ horizontal matchings of distance between them, since $j + i$ is taken modulo r and $j \geq 1$ imply that $i \leq r - 1$.

It is easy to verify that the $r - 1$ horizontal matchings of distance linking two given rows are pairwise disjoint. The next result will be relevant to determine disjoint groups of pairs of rows of

$K_{r \times p}$. We emphasize that we will mention complete graphs with a quantity of vertices that alludes to the p rows of $K_{r \times p}$. When this happens, we will denote the matchings by P_i instead of M_i , the usual notation for matchings. Let K_n be a complete graph with $n \geq 4$ even. Such graph has $n - 1$ disjoint matchings. This result is due to [Soifer, 2008]. Observe Figure 3, where the matchings of K_8 were obtained by Soifer's result. With these preliminary observations, we are ready to determine the equitable total chromatic number of $K_{r \times p}$ for $r \geq 4$ even and p even.

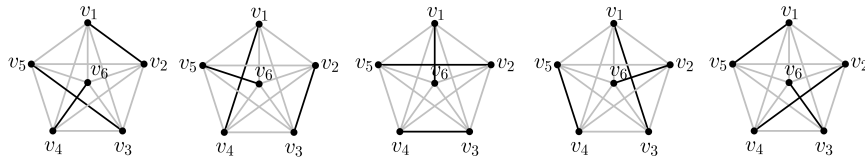


Figure 3: Matchings of K_6

A *Latin square* is an $n \times n$ matrix whose entries are the elements of the set $\{1, 2, \dots, n\}$ such that each symbol occurs precisely once per row and per column. Given a Latin square of order n , a *transversal* is a set of n different entries of different rows and columns. McKay, McLeod and Wanless (2006) proved the following theorem: defining $T(n)$ as the maximum number of transversals over all Latin squares of order n , we have that $b^k \leq T(k)$ for $k \geq 5$, where $b \approx 1,719$. We present now a lemma and omit its proof, because it is trivial.

Lemma 1 *There exists a Latin square of even order $n \geq 4$ whose elements in the main diagonal are pairwise different.*

Algorithm for the case $p = 2$:

First step: obtain a Latin square of order r whose main diagonal entries are all distinct to color vertices and non horizontal edges. We define a *coloring matrix* as a matrix whose entries determine the colors assigned to the elements of a graph. Let $A_{R_1 R_2}$ be a matrix of order r in which the entry a_{ij} represents the color that the edge $x_{i1}x_{j2}$ receives if $i \neq j$ and the color that the vertices of the part X_i receive, otherwise. The matrix $A_{R_1 R_2}$ must be a Latin square whose elements in the main diagonal are all distinct. Lemma 1 ensures the existence of such matrix. Each one of these r colors is used $r + 1$ times. For $K_{4 \times 2}$, for instance, we have

Second step: color horizontal edges with the matchings of K_r . Since $\Delta + 1 = 2r - 1$ in this case, $r - 1$ colors still need to be used. They will be applied in horizontal edges as follows: obtain the $r - 1$ matchings of K_r . Suppose that $R_i = \{v_{a_1}v_{a_2}, \dots, v_{a_{r-1}}v_{a_r}\}$. Then the edges $x_{a_1 1}x_{a_2 1}, \dots, x_{a_{r-1} 1}x_{a_r 1}, x_{a_1 2}x_{a_2 2}, \dots, x_{a_{r-1} 2}x_{a_r 2}$ must receive the same color. This must be done for each $i = 1, 2, \dots, r - 1$. This group of colors is applied on a perfect matching of $K_{r \times 2}$, that is, on r edges. By construction we get that the algorithm describes an equitable total coloring of $K_{r \times 2}$ ($r \geq 4$ even). The matchings of K_4 are $R_1 = \{v_1v_2, v_3v_4\}$, $R_2 = \{v_2v_3, v_1v_4\}$ and $R_3 = \{v_1v_3, v_2v_4\}$. Therefore, the horizontal edges are to be colored as shown in Figure ??.

Algorithm for the case $p = 4$:

First step: coloring the vertices and horizontal edges. To color the vertices we use a different color for each one of the following pairs: x_{11} and x_{12} ; x_{13} and x_{14} ; x_{21} and x_{22} ; x_{23} and x_{24} ; $\dots$; $x_{(r-1)1}$ and $x_{(r-1)2}$; $x_{(r-1)3}$ and $x_{(r-1)4}$; x_{r1} and x_{r4} ; x_{r2} and x_{r2} . The part X_r is the only one that has a different pattern for the coloring of the vertices. Taking $K_{4 \times 4}$, for example, we have to determine matchings of K_4 , which are: $R_1 = \{v_1v_2, v_3v_4\}$, $R_2 = \{v_2v_3, v_1v_4\}$ and

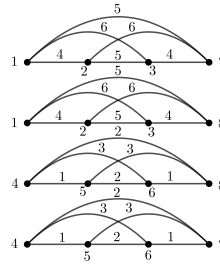


Figure 4: Coloring vertices and horizontal edges of $K_{4 \times 4}$

$R_3 = \{v_1v_3, v_2v_4\}$. We use the colors of vertices of parts $X_1, X_2, \dots, X_{r-1}$ to color horizontal edges. The coloring of vertices and horizontal edges is pictured in Figure 4.

Second step: coloring non horizontal edges of matrices $A_{R_1R_3}$ and $A_{R_2R_4}$. The colors used in the vertices of the rows 1 and 2 of the parts $X_1, X_2, \dots, X_{r-1}$ will be applied on rows 3 and 4 in horizontal edges according to the matchings of K_r , whereas the colors used in the vertices of rows 3 and 4 of the same parts will be assigned to horizontal edges of rows 1 and 2 according to the matchings of K_r . If $R_1 = \{v_{b_1}v_{b_2}, \dots, v_{b_{r-1}}v_{b_r}\}$, then assign the color of the vertices x_{11} and x_{12} to the edges $x_{b_1i}x_{b_2i}, \dots, x_{b_{r-1}i}x_{b_{r-1}i}$ ($i = 3, 4$) and assign the color of the vertices x_{13} and x_{14} to the edges $x_{b_1i}x_{b_2i}, \dots, x_{b_{r-1}i}x_{b_{r-1}i}$ ($i = 1, 2$). Proceed analogously regarding the colors of the vertices of the parts $X_2, X_3, \dots, X_{r-1}$.

Like in the case $p = 2$, we define matrices $A_{R_1R_2}, A_{R_3R_4}, A_{R_2R_3}, A_{R_1R_4}, A_{R_1R_3}$ and $A_{R_2R_4}$, of order r , where the entry a_{ij} of the matrix $A_{R_kR_l}$ represents the color that the edge $x_{ik}x_{jl}$ receives if $i \neq j$. The difference is that here we leave the entry empty if $i = j$.

We have to use $r - 1$ colors in matrices $A_{R_1R_3}$ and $A_{R_2R_4}$, according to the following pattern: on the first row, we apply the colors in ascending order, that is $(1 \ 2 \ 3 \ \dots \ r)$; on the second row, we shift the first row one unity to the right, that is, $(r \ 1 \ \dots \ (r - 1))$. After that, we omit the entries of the main diagonal. By the third step of the algorithm we obtain, for $K_{4 \times 4}$,

the following matrices: $A_{R_1R_3} = A_{R_2R_4} = \begin{bmatrix} & 9 & 10 & 11 \\ 11 & & 9 & 10 \\ 10 & 11 & & 9 \\ 9 & 10 & 11 & \end{bmatrix}$

Fourth step: filling the entries of $A_{R_1R_2}$ and $A_{R_3R_4}$. To fill the matrices $A_{R_1R_2}$ and $A_{R_3R_4}$ we need Latin squares with the elements in the main diagonal being pairwise different (even though such entries will be omitted). Lemma 1 ensures the existence of such Latin squares. The entries of the matrix $A_{R_1R_2}$ will be the colors used in the vertices of rows 1 and 2 of parts $X_1, X_2, \dots, X_{r-1}$, whereas the entries of the matrix $A_{R_3R_4}$ are the colors used in the vertices of rows 3 and 4 of the same parts. Since the matrices have order r , but at this step they are going to be filled with $r - 1$ colors, some entries stay empty at this point. Obtain a Latin square whose colors are described above and whose entries of the main diagonal are all distinct. The empty entries are the ones that would be the same color of the entry a_{rr} . For example, taking the Latin square

$$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix} \text{ for } A_{R_1R_2} \text{ and } A_{R_3R_4} \text{ we get: } A_{R_1R_2} = \begin{bmatrix} & 3 & * & 2 \\ * & & 1 & 3 \\ 2 & * & & 1 \\ 3 & 1 & 2 & \end{bmatrix}; A_{R_3R_4} = \begin{bmatrix} & 6 & * & 5 \\ * & & 4 & 6 \\ 5 & * & & 4 \\ 6 & 4 & 5 & \end{bmatrix}$$

Fifth step: applying colors of vertices x_{r2} and x_{r2} in matrices $A_{R_2R_3}$ and $A_{R_1R_4}$. Suppose that colors α and β have been applied, respectively, in the vertices x_{r1} and x_{r4} ;

and in the vertices x_{r2} and x_{r3} . Then, in the matrix $A_{R_1 R_4}$, color α must occupy entries $a_{1,r-1}$, a_{21} , a_{32} , a_{43} , $\dots$, $a_{r-1,r-2}$, whereas color β must be occupy entries a_{12} , a_{23} , a_{34} , $\dots$, $a_{r-1,r}$, a_{r1} . In matrix $A_{R_2 R_3}$, the corresponding entries occupied with α in the other matrix must be filled with β and vice versa. It is easy to see that we can apply $r - 4$ colors in horizontal matchings of distance linking vertices of rows 1 and 4; and rows 2 and 3. By the fifth step, it follows that

$$A_{R_2 R_3} = \begin{bmatrix} 7 & 8 & * \\ 8 & 7 & * \\ & 8 & 7 \\ 7 & * & * \end{bmatrix}; A_{R_1 R_4} = \begin{bmatrix} 8 & 7 & * \\ 7 & 8 & * \\ & 7 & 8 \\ 8 & * & * \end{bmatrix}.$$

Sixth step: using the last two colors. Up to this point, there are entries in the matrices $A_{R_1 R_2}$, $A_{R_3 R_4}$, $A_{R_2 R_3}$ and $A_{R_1 R_4}$ not filled with any color. Such entries represent edges that form a 2-regular subgraph H of $K_{r \times p}$. Since H is 2-regular, its connected components are cycles. It can be easily seen that none of the connected components is a cycle odd size. Since the components of H are cycles of even size, these edges can be colored with 2 colors and this finishes the algorithm. Finally, by the sixth step, we use colors 13 and 14 to color the edges that have not received a color yet, which concludes the coloring of $K_{4 \times 4}$.

Algorithm for the case $p \geq 6$:

First step: vertex coloring. To color the vertices, we need to obtain the matching P_1 of the graph K_p . If $P_1 = \{v_{b_1}v_{b_2}, v_{b_3}v_{b_4}, \dots, v_{b_{p-1}}v_{b_p}\}$, then assign a different color to each one of the following pairs of vertices: x_{ib_1} and x_{ib_2} ; x_{ib_3} and x_{ib_4} ; $x_{ib_{p-1}}$ and x_{ib_p} for all $i = 1, 2, \dots, r$.

Observe in Figure 5 the scheme to color the vertices of $K_{4 \times 6}$. We have that the matching P_1 of K_6 is $P_1 = \{v_1v_2, v_3v_5, v_4v_6\}$.

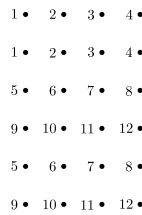


Figure 5: Scheme of vertex coloring of $K_{4 \times 6}$

Second step: using the colors of vertices in non horizontal edges. Consider the matrices $A_{R_{b_1} R_{b_2}}, A_{R_{b_3} R_{b_4}}, \dots, A_{R_{b_{p-1}} R_{b_p}}$ as described in the beginning of this section. We use Lemma 1 to get Latin squares whose entries in the main diagonal are pairwise distinct. Furthermore, the entry a_{kk} of a matrix $A_{R_i R_j}$ must be the colors of the vertices x_{ki} and x_{kj} . The entries of the matrices $A_{R_{b_i} R_{b_j}}$ are the colors used in the vertices of rows b_i and b_j . By the second step of the algorithm applied to graph $K_{4 \times 6}$, we can choose

$$A_{R_1 R_2} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}. \text{ We take } A_{R_3 R_5} = \begin{bmatrix} 5 & 7 & 8 & 6 \\ 8 & 6 & 5 & 7 \\ 6 & 8 & 7 & 5 \\ 8 & 5 & 6 & 8 \end{bmatrix} \text{ and } A_{R_4 R_6} = \begin{bmatrix} 9 & 11 & 12 & 10 \\ 12 & 10 & 9 & 11 \\ 10 & 12 & 11 & 9 \\ 11 & 9 & 10 & 12 \end{bmatrix}.$$

Third step: represent colors $1, 2, \dots, \frac{rp}{2}$ in vertices they had not been used yet. At the end of the second step, colors $1, 2, \dots, \frac{rp}{2}$ were represented in all vertices of two rows. However, they still need to be represented in the vertices of the other rows. We need the following result to do so: [Alspach e Gavlas, 2001] for positive even integers m and n with $4 \leq m \leq n$, the graph $K_n - I$ can be decomposed into cycles of size m if and only if the number of edges in $K_n - I$ is a multiple

of m . We remark that $K_n - I$ denotes a complete graph with n vertices minus a perfect matching.

For the next step of the algorithm we need to get $\frac{p}{2}$ cycles of size $p - 2$ of the graph K_p minus a 1-factor. Putting $m = p - 2$ and $n = p$ in Alspach and Gaylas' result, we conclude that $K_p - I$ can be decomposed into $\frac{p}{2}$ cycles of size $p - 2$, as desired.

Suppose that $K_p - I = K_p \setminus P_1$, with P_1 being a perfect matching of K_p . It is known that every cycle of even size has an edge coloring with 2 colors. Hence we divide each cycle of even order in two matchings and associate with the edges of P_1 , so that each edge of P_1 is associated to the matchings of the cycle of $K_p - I$ that does not contain the vertices v_i and v_j .

With the decomposition of $K_p \setminus P_1$, we get $\frac{p}{2}$ cycles. Let M_k and M'_k be the matchings obtained from the k -th cycle of the decomposition of $K_s \setminus P_1$, that does not contain the edge $v_i v_j$. Then, the colors used in the i -th and j -th vertices of the parts $X_1, X_2, \dots, X_{\frac{r}{2}}$ must be used in horizontal matchings of distance linking vertices of rows determined by M_k , whereas the colors used in the i -th and j -th vertices of parts $X_{\frac{r}{2}+1}, \dots, X_r$ are used in horizontal matchings of distance linking vertices of rows determined by M'_k .

By the third step, we take $(v_3 v_4 v_5 v_6 v_3)$, $(v_2 v_4 v_1 v_6 v_2)$ and $(v_1 v_3 v_2 v_5 v_1)$ as decomposition of $K_6 - \{v_1 v_2, v_3 v_5, v_4 v_6\}$ and we associate, respectively, such cycles with $v_1 v_2$, $v_3 v_5$ and $v_4 v_6$. Thus, we must apply colors 1 and 2 in horizontal matchings of distance 1 and 2, respectively, in edges that link vertices of rows 3 and 4; and 5 and 6. We will represent the application of the 12 colors that we used so far in Table 1.

Table 1: Application of colors 1, 2, $\dots$, 12 in horizontal matchings of distance

Color	Distance	Combination of rows
1	1	3 and 4; 5 and 6
2	2	3 and 4; 5 and 6
3	1	4 and 5; 3 and 6
4	2	4 and 5; 3 and 6
5	1	2 and 4; 1 and 6
6	2	2 and 4; 1 and 6
7	1	4 and 1; 2 and 6
8	2	4 and 1; 2 and 6
9	1	1 and 3; 2 and 5
10	2	1 and 3; 2 and 5
11	1	3 and 2; 1 and 5
12	2	3 and 2; 1 and 5

Fourth step: using $\frac{r}{2} - 1$ colors in perfect matchings of $K_{r \times p}$. There are still $\frac{r}{2} - 1$ horizontal matchings of distance available in each matching P_i ($2 \leq i \leq r$). In other words, $(\frac{r}{2} - 1)(p - 2)$ colors can still be applied in these available matchings of distance. Applying the fourth step to graph $K_{4 \times 6}$, it follows that we still have a horizontal matching of distance 3 for P_2, P_3, P_4 and P_5 . Thus, we use colors 13, 14, 15 and 16 for each one of the matchings of distance 3 linking vertices of rows determined by P_2, P_3, P_4 and P_5 .

Fifth step: coloring horizontal edges. To end the coloring, we use $r - 1$ new colors in horizontal edges determined by the matchings of K_r as follows: if $R_1 = \{v_{a_1} v_{a_2}, v_{a_3} v_{a_4}, \dots, v_{a_{r-1}} v_r\}$, then we apply one of the new colors in the edges $x_{a_1 i} x_{a_2 i}, x_{a_3 i} x_{a_4 i}, \dots, x_{a_{r-1} i} x_{a_r i}$ for all $i = 1, 2, \dots, p$. Using one different color to each matching R_i of K_r , we conclude that $r - 1$ colors are used in this step, as we claimed above. Since $r = 4$ in the example we are coloring, to apply

the fifth step we need to obtain the matchings of K_4 , which are: $R_1\{v_1v_2, v_3v_4\}$, $R_2\{v_2v_3, v_1v_4\}$ and $R_3\{v_1v_3, v_2v_4\}$. Thus, we use color 17 in edges $x_{1i}x_{2i}$ and $x_{3i}x_{4i}$ for all $i = 1, 2, \dots, 6$. Analogously, we use color 18 in edges $x_{2i}x_{3i}$ and $x_{1i}x_{4i}$ and color 19 in edges $x_{1i}x_{3i}$ and $x_{2i}x_{4i}$ for all $i = 1, 2, \dots, 6$.

4. Final considerations and publications

In [da Silva, 2018] there is a study on list edge coloring, a particular type of edge coloring, and new results on equitable total coloring of graphs. Such results were presented at Latin-American Workshop on Cliques in Graphs (LAWCG) in 2016 [da Silva et al., 2017]. A complete version of the paper was published at the Discrete Applied Mathematics (DAM) [da Silva et al., 2018]. Some of the new results were submitted and accepted for presentation at the conference Latin and American Algorithms, Graphs and Optimization Symposium (LAGOS) [da Silva et al., 2019b], that will be held in Brazil, in 2019, and were presented at the conference Cologne Twente Workshop (CTW) in 2018. Such results were recently submitted to the journal Discrete Applied Mathematics. It is important to emphasize that the new results were developed in collaboration with Dr. Simone Dantas and Dr. Diana Sasaki. It is worthy noting that we recently submitted a manuscript to DAM [da Silva et al., 2019a].

References

- Alspach, B. e Gavlas, H. (2001). Cycle decompositions of K_n and $K_n - I$. *Journal of Combinatorial Theory*, 81:77–99.
- Behzad, M. (1965). *Graphs and their chromatic numbers*. PhD thesis, Michigan State University.
- Bondy, J. A. e Murty, U. S. R. (2008). *Graph Theory*. Springer Publishing Company.
- da Silva, A. G. (2018). *Um estudo sobre coloração de arestas e coloração total de grafos*. PhD thesis, Pontifical Catholic University of Rio de Janeiro.
- da Silva, A. G., Dantas, S., e Sasaki, D. (2017). On equitable total coloring of complete r -partite graphs. *Matemática Contemporânea*, 45:143–153.
- da Silva, A. G., Dantas, S., e Sasaki, D. (2018). Equitable total coloring of complete r -partite p -balanced graphs. *Discrete Applied Mathematics*. URL <https://doi.org/10.1016/j.dam.2018.03.009>.
- da Silva, A. G., Dantas, S., e Sasaki, D. (2019a). Determining equitable total chromatic number of infinite classes of complete r -partite graphs. *Submitted to Discrete Applied Mathematics*.
- da Silva, A. G., Dantas, S., e Sasaki, D. (2019b). Equitable total chromatic number of $k_{r \times p}$ for p even. *Accepted at Electronic Notes in Theoretical Computer Science (LAGOS 2019)*.
- Galvin, F. (1995). The list chromatic index of a bipartite multigraph. *Journal of Combinatorial Series*, 63:153–158.
- Soifer, A. (2008). *The Mathematical Coloring Book*. Springer-Verlag.
- Vizing, V. G. (1968). Some unsolved problems in graph theory. *Russian Math. Surveys*, 23(6): 125–141.
- Wang, W. F. (2002). Equitable total coloring of graphs with maximum degree 3. *Graphs Combin.*, 18:677–685.