

MATHEMATICAL PROGRAMMING TECHNIQUES APPLIED TO THE LABEL-CONSTRAINED MINIMUM SPANNING TREE PROBLEM

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RESUMO

O problema da árvore geradora mínima rotulada (PAGMR) é definido em um grafo não-direcionado $G = (V, E, L)$ com conjunto de vértices V , conjunto de arestas E , e conjunto de rótulos L . Cada aresta $e = \{i, j\} \in E$ é associada a um custo não-negativo c_e e a um rótulo $l_e \in L$. O PAGMR é NP-Difícil e visa determinar uma árvore geradora mínima com no máximo K rótulos distintos, onde K é um número inteiro positivo dado como entrada. Neste trabalho, propomos modelos em programação matemática para o PAGMR, assim como um algoritmo de decomposição de Benders. Experimentos realizados em instâncias aleatórias indicam que nossas abordagens alcançam melhores resultados do que os encontrados na literatura. Para valores pequenos de K , em particular, o algoritmo de Benders se mostra a mais promissora das técnicas.

PALAVRAS CHAVE. Otimização combinatória, Programação matemática, Problema da árvore geradora mínima rotulada.

Área Principal: Otimização combinatória, Programação matemática.

ABSTRACT

The label-constrained minimum spanning tree (LCMST) problem is defined for an undirected graph $G = (V, E, L)$ with set of vertices V , set of edges E , and set of labels L . Each edge $e = \{i, j\} \in E$ is associated with a non-negative cost c_e and a label $l_e \in L$. The LCMST problem is NP-Hard and aims at determining a minimum spanning tree with at most K distinct labels, where K is a given positive integer. In this work, we propose mathematical programming models for finding the LCMST as well as a Benders decomposition algorithm. Experiments performed on randomly generated instances show that our exact approaches outperform those from the literature. For small values of K , in particular, the Benders algorithm shows to be the most promising exact technique.

KEYWORDS. Combinatorial optimization, Mathematical programming, Label-constrained minimum spanning tree.

Main Area: Combinatorial optimization, Mathematical programming.

1. Introduction

Some network design problems require integrating distinct types of data transmission equipments as coaxial cable, fiber optics cable, twisted pairs cable, microwave, etc. Communication along each link requires a specified media type. Smaller is the number of different media types in a network topology, less complex is its data communication process. Other important and frequent concern for network designers is the cost of the data communication network defined as the sum of its link costs. In this work we are interested in a tree network topology of minimum cost where the number of distinct media types allowed for the whole network is limited above by a given threshold.

Formally, we are given an undirected connected graph $G = (V, E, L)$ with set of vertices V , set of links (edges) E , and set of labels (distinct media types) L . Each edge $e = \{i, j\} \in E$ is associated with a non-negative cost c_e and a label $l_e \in L$. The problem is to determine a label-constrained minimum spanning tree (LCMST, for short) of G where the maximum number of distinct types of communications media is at most a given integer K . The LCMST problem was introduced and proved to be NP-Hard in (Xiong, 2005). In the literature, we find mixed integer linear programming (MILP) models for finding the LCMST, such as flow-based ones (Xiong et al., 2008). Heuristic methodologies can also be found for the problem such as the genetic algorithm (GA) in (Xiong et al., 2008) and the variable neighborhood search (VNS) in (Naji-Azimi et al., 2010). In the former approach, each GA chromosome is encoded as a set of labels from the total set of combinations $2^{|L|}$. The decoding stage is conducted by Prim's algorithm, which computes a minimum spanning tree in the subgraph induced only by labels available in the chromosome. In the latter approach, an initial solution is constructed by VNS when selecting a minimal set of labels that results in a connected graph. Both shaking and local search phases follow the VNS paradigm and dynamically explore distant neighborhoods of the current incumbent solution.

In this paper, we propose and compare several mathematical programming models and a Benders decomposition algorithm for the LCMST problem. Computational experiments performed on randomly generated instances indicate that our approaches outperform state-of-the-art MILP models for the problem. In particular, for small values of K (for instance, $K \leq 8$), we show that the Benders algorithm is more efficient than current exact solution approaches.

The remainder of the text is organized as follows. In Section 2, we present existing models for the LCMST problem. In Section 3, we conceive new mathematical programming models. In Section 4, we outline a Benders decomposition algorithm. In Section 5, we present computational experiments. In Section 6, we present a brief conclusion and further directions of research.

2. Flow-based models

Initially, we present two flow-based MILP models for the LCMST problem from the literature (Xiong et al., 2008). Consider the following binary decision variables.

$$x_e = \begin{cases} 1, & \text{if edge } e \text{ belongs to a spanning tree solution,} \\ 0, & \text{otherwise.} \end{cases}$$

for all $e \in E$, and

$$y_k = \begin{cases} 1, & \text{if label } k \text{ is used in a spanning tree solution,} \\ 0, & \text{otherwise.} \end{cases}$$

for all $k \in L$.

2.1. Aggregated single-commodity flow-based model

The single-commodity flow-based model uses flow variables $f_{ij} \geq 0$, for all $(i, j) \in A$, where A denotes the set of arcs obtained as $A = \{(i, j), (j, i) : \forall \{i, j\} \in E\}$, with each arc

$(i, j) \in A$ having label l_{ij} equivalent to l_e , $e = \{i, j\}$. The idea is to set vertex 1 as root node. It sends one unit of flow to each other node. The model is as follows.

$$(SCF) \quad \min \quad \sum_{e \in E} c_e x_e \quad (1)$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = |V| - 1 \quad (2)$$

$$\sum_{(i,j) \in A} f_{ij} - \sum_{(j,i) \in A} f_{ji} = 1, \quad \forall j \in V \setminus \{1\} \quad (3)$$

$$\sum_{(i,1) \in A} f_{i1} - \sum_{(1,i) \in A} f_{1i} = -(|V| - 1) \quad (4)$$

$$f_{ij} + f_{ji} \leq (|V| - 1)x_e, \quad \forall e = \{i, j\} \in E \quad (5)$$

$$\sum_{e \in E \mid l_e = k} x_e \leq (|V| - 1)y_k, \quad \forall k \in L \quad (6)$$

$$\sum_{k \in L} y_k \leq K \quad (7)$$

$$x_e \in \{0, 1\}, \quad \forall e \in E \quad (8)$$

$$y_k \in \{0, 1\}, \quad \forall k \in L \quad (9)$$

$$f_{ij} \geq 0, \quad \forall (i, j) \in A \quad (10)$$

The objective function (1) minimizes the cost of the spanning tree. Constraint (2) limits the number of edges to be $|V| - 1$. Constraints (3) and (4) ensure that the root vertex “1” sends one unit of flow for the remaining $V \setminus \{1\}$ vertices. Constraints (5) establish that the flow transits only on edges belonging to the spanning tree. Constraints (6) state that we can have up to $(|V| - 1)$ edges with label $k \in L$ in the solution if this label is used. Constraint (7) sets an upper bound on the number of distinct labels. Constraints (8)–(10) define the domain of the decision variables.

2.2. Multi-commodity flow-based model

Define the following decision variables.

$$a_{ij} = \begin{cases} 1, & \text{if arc } (i, j) \text{ belongs to a spanning tree solution,} \\ 0, & \text{otherwise.} \end{cases}$$

for all $(i, j) \in A$. In addition, let $f_{ij}^h \geq 0$ be the flow from the root node 1 to vertex h in arc (i, j) , for all $(i, j) \in A$ and for all $h \in V \setminus \{1\}$. The model is the following.

$$(MCF) \quad \min \quad \sum_{e \in E} c_e x_e$$

$$\text{s.t.} \quad (2), (6), (7), (8), (9)$$

$$\sum_{(i,j) \in A} f_{ij}^j - \sum_{(j,i) \in A} f_{ji}^j = 1, \quad \forall j \in V \setminus \{1\} \quad (11)$$

$$\sum_{(i,1) \in A} f_{i1}^j - \sum_{(1,i) \in A} f_{1i}^j = -1, \quad \forall j \in V \setminus \{1\} \quad (12)$$

$$\sum_{(i,j) \in A} f_{ij}^h - \sum_{(j,i) \in A} f_{ji}^h = 0, \quad \forall h, j \in V \setminus \{1\}, j \neq h \quad (13)$$

$$f_{ij}^h \leq a_{ij}, \quad \forall (i, j) \in A, \forall h \in V \setminus \{1\} \quad (14)$$

$$a_{ij} + a_{ji} \leq x_e, \quad \forall e = \{i, j\} \in E \quad (15)$$

$$a_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A \quad (16)$$

$$f_{ij}^h \geq 0, \quad \forall (i, j) \in A, \forall h \in V \setminus \{1\} \quad (17)$$

Constraints (11)–(13) are flow conservation constraints guaranteeing a connected solution. Constraints (14) and (15) establish that if some flow is sent along an arc, then the corresponding edge must be selected in the tree. Constraints (16) and (17) define the domain of decision variables.

3. New MILP models

In this section, we formulate the LCMST problem by employing well-known sub-tour elimination constraints (SECs). The root vertex is the one labeled as “1”. For convenience, we adopt the same decision variables defined in Section 2, namely, $x \in \{0, 1\}^{|E|}$, $a \in \{0, 1\}^{|A|}$, and $y \in \{0, 1\}^{|L|}$. The first model is

$$\begin{aligned} (EDM) \quad & \min \sum_{e \in E} c_e x_e \\ & \text{s.t.} \quad (2), (6), (7), (8), (9) \\ & \quad \sum_{e \in E(S)} x_e \leq |S| - 1, \quad \forall S \subset V : |S| > 2 \end{aligned} \quad (18)$$

Constraints (18) are SECs avoiding cycles in any $S \subset V$. This model has an exponential number of such constraints. To overcome using all of them, we resort to a branch-and-cut framework as will be further detailed in Section 5.

3.1. Miller–Tucker–Zemlin constraints

The model of this section takes into account the well-known Miller–Tucker–Zemlin constraints (Miller et al., 1960) to avoid cycles by setting labels to vertices belonging to the solution. Let $u_i \geq 0$ be continuous variables, for all $i \in V$. The model is

$$(MTZ) \quad \min \sum_{(i,j) \in A} c_{ij} a_{ij} \quad (19)$$

$$\text{s.t.} \quad (7), (9), (16)$$

$$\sum_{(i,j) \in A} a_{ij} = |V| - 1 \quad (20)$$

$$\sum_{i \in V \mid (i,j) \in A} a_{ij} = 1, \quad \forall j \in V \setminus \{1\} \quad (21)$$

$$u_i - u_j + |V| a_{ij} \leq |V| - 1, \quad \forall (i, j) \in A \quad (22)$$

$$\sum_{(i,j) \in A \mid l_{ij}=k} a_{ij} \leq (|V| - 1) y_k, \quad \forall k \in L \quad (23)$$

$$u_1 = 0 \quad (24)$$

$$u_i \geq 0, \quad \forall i \in V \quad (25)$$

The objective function (19) minimizes the cost of the spanning tree defined by arcs. Constraint (20) imposes that the number of arcs in the spanning tree is $|V| - 1$. Constraints (21) guarantee that the number of arcs entering vertex $j \in V - \{1\}$ is equal to one. Constraints (22) are SECs assigning labels to vertices such that the label of vertex j is at least one unit more than the one of vertex i if arc (i, j) is in the solution. Constraints (23) have the same interpretation as in (6). Constraints (24) and (25) define the domain of decision variables.

3.2. Directed cutset constraints

Directed cutset constraints Magnanti and Wolsey (1995) are often employed into exact methods to avoid cycles in any integer solution. The model is as follows.

$$\begin{aligned}
 (DCUT) \quad & \min \sum_{(i,j) \in A} c_{ij} a_{ij} \\
 \text{s.t.} \quad & (7), (9), (16), (20), (21), (23) \\
 & \sum_{(i,j) \in A \mid j \in S} a_{ij} \geq 1, \quad S \subset V : \{1\} \not\subset S
 \end{aligned} \tag{26}$$

Constraints (26) ensure that for any subset of vertices $S \subset V$, $\{1\} \not\subset S$, there is at least one arc going from $V - S$ to S . As for model (EDM), it is not possible to enumerate all $S \subset V$. Consequently, they are also handled in a branch-and-cut framework.

3.3. Kaibel–Pashkovich–Theis based constraints

Let an arbitrary edge $e = \{u, v\} \in E$ be part of a LCMST integer solution. When removing e from it, each vertex w in $V \setminus \{u, v\}$ must be either in the component of u or in the one of v . If w belongs to both components, then it is clear that u, v , and w induce a cycle in the foregoing (infeasible) solution. The idea presented in (Kaibel et al., 2012) is to define three-index variables $z_{ij}^e \geq 0$, where $z_{ij}^e = 1$ if edge $e = \{u, v\}$ belongs to the spanning tree and vertex j is in the component of vertex i , for all $e \in E$, i being extremity of $e = \{u, v\}$, and $j \in V \setminus \{u, v\}$. The LCMST model is

$$\begin{aligned}
 (KPT) \quad & \min \sum_{e \in E} c_e x_e \\
 \text{s.t.} \quad & (2), (6), (7), (8), (9) \\
 & x_e - z_{ik}^e - z_{jk}^e = 0, \quad \forall e = \{i, j\} \in E, \forall k \in V \setminus \{i, j\} \tag{27} \\
 & x_{\{i,j\}} + \sum_{h=\{i,k\} \in E \mid k \notin \{i,j\}} z_{kj}^h = 1, \quad \forall i \in V, \forall j \in V - i \mid \{i, j\} \in E \tag{28} \\
 & z_{ij}^e \in \{0, 1\}, \quad \forall e = \{u, v\} \in E, \forall i \in \{u, v\}, \forall j \in V \setminus \{u, v\} \tag{29}
 \end{aligned}$$

Constraints (27) and (28) break sub-tours by ensuring that if an edge $e = \{u, v\}$ belongs to the spanning tree, than the vertices in $V \setminus \{u, v\}$ may be either on the component of u or in the component of v . Constraints (29) define the domain of decision variables.

3.4. Martin based constraints

The model of this section uses decision variables presented in (Martin, 1991) to break cycles. We defined $\lambda_{ij}^k = 1$ if edge $e = \{i, j\}$ is in the solution (i.e. $x_e = 1$) and it is oriented with respect to node k from j to i in the graph induced by x ; and 0 otherwise, for all $k \in V$, and for all $e = \{i, j\} \in E$. The model is as follows.

$$\begin{aligned}
 (MRT) \quad & \min \sum_{e \in E} c_e x_e \\
 \text{s.t.} \quad & (2), (6), (7), (8), (9) \\
 & \lambda_{ij}^k + \lambda_{ji}^k = x_e, \quad \forall k \in V, \forall e = \{i, j\} \in E \tag{30} \\
 & \sum_{\{i,j\} \in E} \lambda_{ij}^k = 1, \quad \forall i, k \in V, i \neq k \tag{31} \\
 & \lambda_{kj}^k = 0, \quad \forall k, j \in V, j \neq k, \mid \{k, j\} \in E \tag{32} \\
 & \lambda_{ij}^k, \lambda_{ji}^k \in \{0, 1\}, \quad \forall k \in V, \forall e = \{i, j\} \in E \tag{33}
 \end{aligned}$$

Constraints (30) and (31) help to prevent the existence of sub-tours in the solution. Constraints (32) and (33) define the domain of decision variables.

3.5. Path orienteering constraints

Let precedence variables $p_{ij} \geq 0$ indicate if there is a directed path going from i to j (value 1) or not (value 0), for all $i, j \in V, i \neq j$. Precedence variables have been recently attached in SECs to solve combinatorial optimization problems defined under a tree topology, e.g. the optimum communication spanning tree (Luna-Mota, 2015). The model is

$$\begin{aligned}
 (PREC) \quad & \min \sum_{(i,j) \in A} c_{ij} a_{ij} \\
 \text{s.t.} \quad & (7), (9), (16), (20), (21), (23) \\
 & a_{ij} \leq p_{ij}, \quad \forall (i, j) \in A \quad (34) \\
 & p_{ij} + p_{ji} \leq 1, \quad \forall i, j \in V, i \neq j \quad (35) \\
 & p_{ij} + a_{jk} \leq 1 + p_{ik}, \quad \forall (j, k) \in A, \forall i \in V \setminus \{j, k\} \quad (36) \\
 & p_{ik} + a_{jk} \leq 1 + p_{ij}, \quad \forall (j, k) \in A, \forall i \in V \setminus \{j, k\} \quad (37) \\
 & p_{ij} \geq 0, \quad \forall i, j \in V, i \neq j \quad (38)
 \end{aligned}$$

Constraints (34) state that if there is an arc $(i, j) \in A$ belonging to the solution, then a directed path from i to j must exist. Constraints (35) establish, for all $(i, j) \in A$, that there is a directed path either from i to j or from j to i . Constraints (36) impose that, if there is a directed path from $i \in V$ to $j \in V$ and arc (j, k) belongs to the solution, then there is also a directed path from i to k . Constraints (37) state that, if there is a directed path from $i \in V$ to $k \in V$ and arc (j, k) belongs to the solution, then there is also a directed path from i to j . Constraints (38) define the domain of decision variables.

4. Benders decomposition

Benders decomposition (Benders, 1962) has been widely applied to hard combinatorial optimization problems (see the recent work of Rahmaniani et al. (2017) for a literature review). The idea behind the classical Benders algorithm is to decompose the original problem into a master problem and a subproblem. The master problem contains complicating variables in the sense that, when they are set to a given value, the resulting optimization problem (namely, the subproblem) becomes a linear problem that is significantly easier to solve and for which we can use standard duality theory to perform row generation to be included to the master problem. Both master problem and subproblem are iteratively solved to guide the search process until a stopping criterion is reached.

To solve the LCMST problem by Benders decomposition, we resort to the flow-based model (*SCF*). When we fix values for complicating variables x and y , say $\bar{x}$ and $\bar{y}$, the resulting problem is as follows.

$$\begin{aligned}
 (SBP) \quad & \min \sum_{(i,j) \in A} 0 f_{ij} \quad (39) \\
 \text{s.t.} \quad & \sum_{(i,j) \in A} f_{ij} - \sum_{(j,i) \in A} f_{ji} = 1, \quad \forall j \in V \setminus \{1\} \quad (40) \\
 & \sum_{(i,1) \in A} f_{i1} - \sum_{(1,i) \in A} f_{1i} = -(|V| - 1) \quad (41) \\
 & f_{ij} + f_{ji} \leq (|V| - 1) \bar{x}_e, \quad \forall e = \{i, j\} \in E \quad (42) \\
 & f_{ij} \geq 0, \quad \forall (i, j) \in A \quad (43)
 \end{aligned}$$

whose objective function (39) can also be calculated by solving its dual. Assuming dual variables $\alpha \in \mathbb{R}^{|V|}$, β , and $\gamma \in \mathbb{R}^{|E|}$ related to constraints (40), (41) and (42), respectively, the subproblem

of the Benders decomposition scheme is as follows.

$$(SB_D) \quad \max \quad \sum_{j \in V \setminus \{1\}} \alpha_j - (|V| - 1)\beta + \sum_{e \in E} \bar{x}_e (|V| - 1)\gamma_e \quad (44)$$

$$\text{s.t.} \quad \sum_{k \in V \setminus \{1\}} g(i, j, k) \alpha_k + h(i, j) \beta + \gamma_{\{i, j\}} \leq 0, \quad \forall (i, j) \in A \quad (45)$$

$$\alpha_j \in \mathbb{R}, \quad \forall j \in V \setminus \{1\} \quad (46)$$

$$\beta \in \mathbb{R} \quad (47)$$

$$\gamma_e \leq 0, \quad \forall e \in E \quad (48)$$

$$\text{with } g(i, j, k) = \begin{cases} -1, & \text{if } k = i, \\ 1, & \text{if } k = j, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } h(i, j) = \begin{cases} -1, & \text{if } i = 1, \\ 1, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases} \quad \text{whereas its corresponding}$$

Benders master problem is written as follows.

$$(M) \quad \min \quad \sum_{e \in E} c_e x_e + z \quad (49)$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = |V| - 1 \quad (50)$$

$$\sum_{e \in E_k} x_e \leq (|V| - 1) y_k, \quad \forall k \in L \quad (51)$$

$$\sum_{k \in L} y_k \leq K \quad (52)$$

$$z \geq \sum_{j \in V \setminus \{1\}} \bar{\alpha}_j - (|V| - 1) \bar{\beta} + \sum_{e \in E} x_e^p (|V| - 1) \bar{\gamma}_e, \quad \forall p \in P \quad (53)$$

$$\sum_{j \in V \setminus \{1\}} \bar{\alpha}_j - (|V| - 1) \bar{\beta} + \sum_{e \in E} x_e^q (|V| - 1) \bar{\gamma}_e \leq 0, \quad \forall q \in Q \quad (54)$$

$$x_e \in \{0, 1\}, \quad \forall e \in E \quad (55)$$

$$y_k \in \{0, 1\}, \quad \forall k \in L \quad (56)$$

where P and Q are the sets of extreme points and extreme rays of the Benders subproblem (SB_D) , respectively.

5. Computational experiments

We implemented all mathematical programming techniques of Sections 2, 3 and 4 and compare all of them. They all were coded using CPLEX 12.7 and Java concert technology. Our experiments were performed on a PC Intel Pentium i7, 8×3.60 GHz, 16 GB RAM under Linux Ubuntu 14.05 LTS. Due to high number of exact methods taken into account to evaluation purposes, we set a CPU time limit of 3600 seconds for each instance. Other CPLEX parameters are set as default values.

Note that models (EDM) and $(DCUT)$ require an exponential number of constraints associated with sub-tour elimination process. To solve LCMST instances with these models, we resort to a branch-and-cut algorithm that searches for violated inequalities (18) and (26). More precisely, we first omit such constraints from (EDM) and $(DCUT)$, respectively, and always when an integer solution is found by CPLEX, we detect connected subsets $S \subset V$ containing a cycle by using a depth first search (DFS) routine. We proceed by adding violated inequalities to (EDM) and $(DCUT)$ until the optimal integer solution is found.

The instances adopted in our experiments are complete graphs. As in (Xiong et al., 2008), we generated them as follows. First, we randomly generated $|V|$ vertices in the integer square

$[0, 999] \times [0, 999]$. The cost c_e of each edge $e = \{i, j\} \in E$ is the Euclidean distance between $i \in V$ and $j \in V$, whereas label l_e is chosen randomly from the integer interval $[1, |L|]$. The number of labels $|L|$ is equal to $|V|$. For each configuration of $|V|$, we generated 5 distinct complete graphs labelled with identifiers to distinguish them. The maximum number of labels K allowed in the spanning tree varies for each instance. Supplementary instances involving those ones devised by Xiong et al. (2008) will be used in a future version of this work.

Results in the next tables are grouped by the number of vertices $|V|$. For each instance, we give the identifier ID , the maximum number of different labels K allowed in the solution, and the optimal solution OPT . For each mathematical programming technique, we report the number of subproblems (bb) solved by CPLEX, the elapsed time (cpu) in seconds to obtain the optimal solution, and the integrality (gap) in percentage between the linear relaxed solution and the optimal one (except for the Benders decomposition algorithm, referred to as BD in the sequel). The symbol “n/a” means values not available due to either time or memory limitation. In our experiments, we considered $|V| = |L| = 25$ (Table 1), $|V| = |L| = 50$ (Table 2), and $|V| = |L| = 75$ (Table 3).

Regarding instances with $|V| = |L| = 25$, (EDM) and BD proved to be the most efficient exact approaches. The average CPU time required to obtain the optimal solution for the whole set of instances was 0.14 and 0.37 seconds, respectively, while (SCF) and (MCF) needed 1.30 and 20.29 seconds. The smallest number of subproblems solved by CPLEX, in average, was obtained by (MRT) and it was 3.20. The tightest linear relaxations were presented by (MCF), (KPT), and (MRT), with average gap equal to 20.27 for each one.

Concerning instances with $|V| = |L| = 50$, the two best CPU times, in average, were observed by (EDM) and (MTZ), namely, 6.44 and 16.79 seconds, respectively. The average CPU time of BD was 32.11 seconds. Again, (MCF), (KPT), and (MRT) proved to have the tightest linear relaxations with average gap equal to 39.96 each one. The average number of subproblems solved by (KPT) was 14.47. For at least one instance, models (SCF), (MCF), ($DCUT$), (MRT), and ($PREC$) did not reach the optimal solution within the time limit.

Finally, for instances with $|V| = |L| = 75$, in particular, we considered only the three most promising (concerning runtime) mathematical programming techniques from Tables 1 and 2, namely, (EDM), (MTZ), and BD. When $K \leq 8$, we noticed a strict dominance of BD: it required a considerably smaller CPU time with respect to both (EDM) and (MTZ). It is worth mentioning that this behavior is also observed for small values of K (suppose $K \leq 8$) in Table 2. The results then suggest that BD is intrinsically more suitable for solving instances with such characteristic. When $K \geq 12$, for most instances (EDM) performed better than (MTZ) and BD. None of them found the optimal solution for the whole set of instances within the time limit.

6. Conclusion

The label-constrained minimum spanning tree (LCMST) problem is an NP-Hard combinatorial optimization problem appearing in network design. In this paper, we provided mathematical programming models for the problem and a Benders decomposition algorithm as well. Computational experiments performed on randomly generated instances indicated that our exact approaches outperform existing flow-based models presented in the literature. When dealing with instances requiring a small number of labels, in particular, we noticed a clear dominance of the Benders decomposition algorithm over LCMST exact models. As future work, we plan to investigate the application of a Lagrangian relaxation to the LCMST problem. We strongly believe that the Lagrangian scheme can be efficiently used to find both good lower bounds and promising feasible solutions for the problem. Investigating valid inequalities is also in our scope of research.

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Tabela 1: Results for $|V| = |L| = 25$

Instance			(SCF)			(MCF)			(EDM)			(DCUT)			(MTZ)			(KPT)			(MRT)			(PREC)			BD	
ID	K	OPT	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu
1	3	6478.91	0	0.80	57.31	20	66.96	50.55	278	0.26	57.64	2100	1.88	57.62	31	0.69	57.21	0	7.13	50.55	0	23.02	50.55	23	19.89	51.61	62	0.57
5	4753.81		0	1.11	41.82	10	15.83	32.60	282	0.21	42.27	2508	1.28	42.24	533	0.31	41.69	5	3.82	32.60	7	12.41	32.60	97	8.63	34.05	335	0.22
7	4033.77		1329	1.82	31.43	29	9.83	20.57	455	0.25	31.96	1985	0.79	31.94	1251	0.46	31.28	17	3.64	20.57	16	12.11	20.57	292	10.13	22.27	555	0.31
9	3580.67		0	0.65	22.76	0	3.36	10.52	72	0.10	23.35	27	0.16	23.32	0	0.20	22.58	0	1.11	10.52	0	2.22	10.52	0	1.95	12.44	177	0.22
11	3325.84		0	0.42	16.84	0	1.88	3.67	107	0.07	17.48	1	0.08	17.45	0	0.24	16.65	0	0.69	3.67	0	1.53	3.67	0	1.84	5.73	0	0.12
13	3235.71		0	0.29	14.52	0	1.75	0.98	0	0.02	15.18	1	0.08	15.15	0	0.26	14.33	3	0.49	0.98	23	1.27	0.98	0	1.25	3.10	0	0.08
2	3	6113.06	0	1.28	59.16	13	45.04	51.04	119	0.26	59.77	2359	1.34	60.65	0	0.39	60.05	0	5.58	51.04	0	20.35	51.04	0	6.81	52.46	17	0.51
5	4604.94		143	0.93	45.78	11	20.87	35.01	138	0.19	46.60	2808	1.38	47.77	0	0.35	46.97	0	4.06	35.01	0	11.09	35.01	0	7.82	36.89	50	0.34
7	3743.93		0	0.69	33.31	0	8.14	20.06	66	0.07	34.32	843	0.52	35.75	0	0.49	34.77	0	1.35	20.06	0	3.63	20.06	0	5.94	22.38	119	0.18
9	3333.35		0	0.82	25.10	0	5.90	10.22	34	0.10	26.23	463	0.18	27.84	2148	0.49	26.74	0	0.92	10.22	0	2.13	10.22	356	7.20	12.82	454	0.30
11	3151.61		496	0.80	20.78	0	2.30	5.04	13	0.04	21.97	134	0.15	23.68	2169	0.42	22.51	0	0.53	5.04	0	1.44	5.04	340	4.46	7.79	255	0.31
13	3065.55		1679	0.89	18.56	0	2.69	2.37	162	0.04	19.78	9	0.14	21.54	3812	0.49	20.34	26	0.81	2.37	23	1.67	2.37	367	3.22	5.20	842	0.23
3	3	7542.74	135	1.33	60.49	0	89.40	56.05	811	0.56	61.71	3617	2.64	64.62	93	0.52	64.27	0	5.96	56.05	0	24.27	56.05	116	24.35	58.63	88	0.95
5	5479.86		1533	2.17	45.61	79	58.81	39.51	332	0.39	47.30	15739	9.25	51.30	2087	0.78	50.81	21	8.04	39.51	16	35.02	39.51	313	18.94	43.06	735	0.32
7	4255.60		0	1.19	29.97	13	21.94	22.11	181	0.14	32.13	2220	0.94	37.29	6683	1.57	36.66	5	3.63	22.11	8	14.87	22.11	1993	27.49	26.68	290	0.35
9	3757.36		0	0.50	20.68	0	4.65	11.78	2	0.09	23.13	2082	0.66	28.97	7498	1.05	28.27	0	1.08	11.78	0	3.55	11.78	774	6.58	16.96	81	0.25
11	3534.99		0	0.67	15.69	0	3.11	6.23	3	0.07	18.30	698	0.23	24.50	5641	1.06	23.75	0	0.77	6.23	0	2.52	6.23	775	6.09	11.74	115	0.22
13	3394.91		0	0.40	12.21	0	1.78	2.36	41	0.03	14.93	144	0.08	21.39	6880	1.06	20.61	25	0.48	2.36	0	0.97	2.36	906	7.80	8.09	127	0.18
4	3	7039.53	1236	2.17	63.03	21	95.11	53.09	482	0.35	64.14	1601	1.79	62.15	110	0.51	61.89	0	6.87	53.09	0	32.43	53.09	43	21.80	57.99	195	0.42
5	5021.35		968	1.55	48.17	9	38.62	34.24	160	0.14	49.72	3173	2.47	46.93	1235	0.65	46.57	3	4.77	34.24	3	17.09	34.24	205	11.14	41.11	380	0.43
7	4128.15		591	1.53	36.96	0	11.61	20.01	9	0.11	38.85	2379	0.91	35.45	3697	0.77	35.01	0	1.42	20.01	0	5.05	20.01	794	13.91	28.37	917	0.33
9	3686.30		483	0.77	29.40	0	7.64	10.42	15	0.07	31.52	1158	0.51	27.71	3334	0.58	27.22	0	0.87	10.42	0	2.71	10.42	315	5.49	19.78	409	0.26
11	3457.59		1754	1.15	24.73	150	9.82	4.50	4	0.02	26.99	490	0.20	22.93	2703	0.51	22.41	0	0.76	4.50	0	2.32	4.50	146	4.40	14.48	1607	0.30
13	3311.70		4294	1.69	21.42	15	2.44	0.29	2	0.02	23.77	145	0.07	19.53	3634	1.02	18.99	0	0.36	0.29	0	1.12	0.29	187	4.05	10.71	3266	0.44
5	5853.57		744	1.99	60.54	23	51.39	46.48	234	0.32	61.84	1169	1.09	54.23	0	0.52	53.85	0	5.49	46.48	0	23.58	46.48	0	14.17	47.24	224	1.38
5	4358.33		1522	1.44	47.01	0	9.66	28.12	4	0.08	48.75	448	0.50	38.52	0	0.12	38.02	0	2.81	28.12	0	8.70	28.12	0	3.93	29.14	345	0.51
7	3756.30		795	1.81	38.51	0	9.48	16.60	340	0.27	28.67	178	0.23	28.08	0	0.12	28.08	0	1.71	16.60	0	5.98	16.60	0	3.91	17.78	694	0.32
9	3391.17		765	1.36	31.89	0	3.47	7.61	0	0.04	34.13	52	0.16	20.99	155	0.19	20.34	0	0.73	7.61	0	2.50	7.61	0	2.77	8.93	195	0.28
11	3270.11		4664	2.44	29.37	0	3.23	4.19	7	0.04	31.69	2	0.13	18.07	684	0.20	17.39	0	0.89	4.19	0	2.64	4.19	75	2.76	5.55	3669	0.44
13	3195.24		9039	4.42	27.72	0	1.96	1.95	12	0.02	30.09	0	0.09	16.15	623	0.17	15.45	20	0.88	1.95	0	2.06	1.95	185	1.51	3.34	4758	0.52
Average			1072.33	1.30	34.36	13.10	20.29	20.27	134.17	0.14	35.87	1623.13	1.00	34.15	1839.30	0.54	33.49	4.17	2.59	20.27	3.20	9.34	20.27	276.73	8.67	23.88	698.70	0.37

Tabela 2: Results for $|V| = |L| = 50$

ID	Instance	(SCF)			(MCF)			(EDM)			(DCUT)			(MTZ)			(KPT)			(MRT)			(PREC)			BD	
		bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	gap	bb	cpu	bb	cpu	
1	3	15725.46	3221	24.29	77.43	2682	7.34	78.21	19481	82.76	77.57	2451	9.23	77.46	23	2234.13	70.01	> 3600	70.01	> 3600	68.22	> 3600	70.16	584	1.73	292	1.66
5	5	10548.67	13922	39.30	66.35	4554	9.02	67.52	272960	703.36	66.57	1947	5.20	66.40	27	910.87	55.29	> 3600	55.29	> 3600	53.82	> 3600	56.64	1322	2.47	462	1.56
	7	8418.98	24722	123.96	57.84	6290	10.98	59.30	461709	1216.59	58.11	5195	11.94	57.90	25	317.92	43.99	> 3600	43.99	> 3600	41.87	> 3600	45.42	1503	8.06	3356	3.51
	9	7168.87	56923	302.69	50.49	4005	7.83	52.20	563538	1106.38	50.81	7391	9.72	50.56	13	207.55	34.22	12	1205.04	34.22	57.13	2102.38	40.12	12316	10.04	76946	65.00
11	6400.2	192454	674.28	44.55	1893	2.36	46.46	1300346	1630.59	44.90	14200	17.73	44.62	7	111.89	26.32	5	726.14	26.32	5	360.10	28.68	138895	90.03	37509	18.47	
13	5817.93	245469	614.53	39.00	10	2176.44	18.94	1710	2.02	41.10	280298	235.79	39.38	58127	43.04	39.08	0	66.27	18.94	0	66.27	18.94	37509	18.47	37509	18.47	
2	3	15452.01	2929	30.16	74.28	12794	36.50	74.60	60799	331.01	74.76	65384	94.28	74.60	92	2970.61	68.22	> 3600	68.22	> 3600	68.22	> 3600	70.16	584	1.73	292	1.66
5	5	10634.61	17588	130.31	62.62	5003	15.41	63.09	206153	914.09	63.32	68841	78.11	63.09	33	947.69	53.82	> 3600	53.82	> 3600	53.82	> 3600	56.64	1322	2.47	462	1.56
	7	8447.93	27166	127.98	52.95	13803	17.94	53.53	> 3600		53.83	2686	5.46	53.54	13	380.19	41.87	13	2429.17	41.87	39.58	1541.89	45.42	1503	8.06	3356	3.51
	9	7313.85	55635	259.19	45.65	3489	6.45	46.33	361782	792.45	46.67	4827	17.58	46.34	22	256.97	32.86	18	970.43	32.86	50.46	960.76	36.95	15363	11.76	68746	49.80
11	6691.68	282761	656.78	40.60	9	2729.93	26.62	5200	6.85	41.34	95966	129.80	41.71	15245	25.64	41.35	5	585.28	26.62	60.18	1068.72	31.09	68746	49.80	347784	212.27	
13	6276.97	> 3600	36.67	36.67	15	2699.03	21.77	4454	5.08	37.46	687814	606.70	37.86	29550	29.27	37.47	9	81.86	21.77	13	445.19	21.77	10724	2133.52	66.51	3726	8.06
3	3	13894.81	20000	67.11	67.73	191	0.94	68.13	7069	20.53	72.09	340	3.81	71.91	5	1014.73	64.02	> 3600	64.02	> 3600	64.02	> 3600	66.51	3726	8.06	345	1.43
5	5	10057.97	4132	19.00	55.42	2197	3.15	55.97	29397	49.78	61.44	1261	4.06	61.19	8	411.37	50.29	5	3097.06	50.29	1108	573.41	53.73	345	1.43	5250	6.82
	7	8630.59	66485	170.49	48.05	1622	2.61	48.69	121728	197.18	55.06	4046	5.13	54.77	11	170.13	42.07	7	1183.08	42.07	4399	1445.85	46.08	15388	33.38	37874	42.41
	9	7698.16	103775	386.95	41.75	1612	2.02	42.48	616538	944.52	49.62	8726	13.33	49.30	5	170.90	35.05	5	743.99	35.05	41.16	983.02	39.55	170197	136.52	170197	136.52
11	7048.89	574205	2132.92	36.39	3607	3.67	37.18	922912	1169.36	44.98	16671	12.72	44.63	11	110.50	29.07	15	657.58	29.07	52.42	936.37	33.98	170197	136.52	170197	136.52	
13	6498.89	405856	1297.72	31.00	29	1709.60	23.07	4836	3.47	31.86	476613	520.44	40.32	14796	14.77	39.94	9	372.78	23.07	54.93	713.67	28.48	161832	83.17	161832	83.17	
4	3	15852.07	403	13.94	73.24	1837	5.33	73.90	12169	58.67	74.09	457	5.20	73.97	15	1465.11	67.66	> 3600	67.66	> 3600	67.66	> 3600	69.98	76	0.97	345	1.22
5	5	11002.23	3331	17.83	61.44	1488	1.66	62.39	217540	577.65	62.67	2114	5.68	62.50	11	489.82	53.40	> 3600	53.4	> 3600	53.4	> 3600	56.75	345	1.22	15388	33.38
	7	8659.37	6772	22.80	51.00	556	0.75	52.21	179468	322.51	52.57	1503	3.97	52.36	0	196.16	40.79	3	1297.59	40.79	36.47	239.55	45.05	18388	35.13	18388	35.13
	9	7582.63	119225	571.69	44.05	1060	1.08	45.43	214771	294.93	45.83	12826	17.98	45.59	7	172.82	32.39	7	795.94	32.39	59.69	1054.79	37.25	104823	54.69	104823	54.69
11	6832.77	153548	696.17	37.91	784	0.92	39.44	604697	715.37	39.89	19156	19.66	39.62	0	77.23	24.97	0	795.08	24.97	109.18	2261.84	30.36	104823	54.69	104823	54.69	
13	6350.29	478888	1708.40	33.19	24	2196.18	19.27	229	0.37	34.84	57552	56.18	35.32	21450	22.18	35.03	0	429.05	19.27	98.92	2106.15	25.07	187907	92.60	187907	92.60	
5	3	13699.68	11803	80.02	74.89	5723	15.93	75.27	33736	115.62	73.30	594	5.59	73.12	39	2280.06	64.95	> 3600	64.95	> 3600	64.95	> 3600	68.29	488	1.43	2799	3.40
5	5	9925.75	39682	103.87	65.35	3819	15.69	65.86	76610	232.81	63.15	1758	5.15	62.90	41	972.48	51.63	> 3600	51.63	> 3600	51.63	> 3600	56.23	340	2.33	1940	2.33
	7	7905.91	9611	27.88	56.49	1779	5.00	57.14	99156	238.90	53.73	0	2.88	53.42	0	271.12	39.27	3	1616.56	39.27	38.15	491.38	45.04	1940	2.33	1940	2.33
	9	6712.98	4941	15.03	48.76	407	0.74	49.52	20789	30.65	45.51	0	1.33	45.15	0	51.64	28.48	0	288.66	28.48	96.2	220.63	35.28	2608	3.89	2608	3.89
11	6150.08	56147	203.51	44.07	3	1342.50	21.93	487	0.94	44.90	143648	147.80	40.52	3718	2.18	40.13	0	224.70	21.93	46.85	639.96	29.36	15120	9.55	15120	9.55	
13	5743.76	85447	240.05	40.12	13	818.66	16.41	931	1.04	41.01	22694	20.50	36.32	9634	10.83	35.89	3	312.67	16.41	47.49	620.02	24.36	37509	18.47	37509	18.47	
Average			n/a	51.98		3303.93	6.44	52.91	n/a	n/a	53.40	13163.13	16.79	53.13	14.47	555.48	39.96	n/a	n/a	n/a	n/a	n/a	44.97	48677.80	32.11	48677.80	32.11

Tabela 3: Results for $|V| = |L| = 75$

Instance			(EDM)		(MTZ)		BD	
ID	K	OPT	bb	cpu	bb	cpu	bb	cpu
1	4	19240.90	> 3600		63228	453.45	1367	8.55
	8	12117.32	102506	564.41	27590	168.91	3028	11.37
	12	9555.60	71142	434.25	5633	34.82	10645	22.05
	16	8151.00	3251	10.76	5522	16.30	8041	14.98
	20	7377.20	2124	4.60	23330	26.68	27622	35.74
	24	6896.32	1430	2.74	69100	41.33	115766	73.63
2	4	18264.57	50253	454.59	18235	181.69	1539	7.27
	8	11916.73	67048	813.49	118743	493.50	21947	82.60
	12	9242.58	853	3.57	60201	133.93	54000	98.88
	16	7927.86	411	1.46	183958	242.22	180180	299.70
	20	7237.77	736	2.23	197002	191.60	> 3600	
	24	6835.09	396	1.02	1354187	946.41	> 3600	
3	4	17561.36	46245	892.30	30421	355.99	2865	13.01
	8	10891.21	357296	2640.66	112050	508.75	113354	437.39
	12	8540.82	57448	196.41	41462	134.13	89864	454.65
	16	7367.18	1987	4.83	46565	116.21	1171616	2133.43
	20	6746.08	3113	5.04	64575	105.99	> 3600	
	24	6307.49	1939	2.29	73349	105.16	> 3600	
4	4	18885.48	46685	847.91	40930	492.35	863	6.38
	8	11553.21	66382	667.16	43395	255.80	25676	163.36
	12	9080.06	27554	80.18	31676	100.67	44818	66.04
	16	7802.03	5924	10.94	47969	96.04	54852	50.47
	20	7003.72	643	1.87	98348	128.94	118527	126.96
	24	6586.09	674	1.76	229001	178.42	1216297	1120.89
5	4	18686.77	55769	878.08	61021	574.51	875	5.52
	8	11951.81	68492	718.60	102418	538.93	186436	787.44
	12	9621.17	51095	182.74	> 3600		> 3600	
	16	8289.25	21226	42.65	393819	461.44	> 3600	
	20	7396.24	1736	3.99	2312093	3487.17	> 3600	
	24	6906.31	44738	42.60	> 3600		> 3600	
Average			n/a		n/a		n/a	

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