Effects of Process Mean Estimation on the $\bar{X}$ Control Chart and Adjustments for a Guaranteed In-Control Performance

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Abstract

The $\bar{X}$ chart is one of the most used tools for monitoring the mean of a process. To design this chart the in-control process mean (μ_0) or standard deviation (σ_0) must be estimated from m historical samples. In this case, the in-control performance of the chart is affected. To study this effect, a recent idea emerged where control charts performance is measured by the probability of the conditional in-control average run length ($CARL_0$) being greater than a given specified value. This is called the Exceedance Probability Criterion (EPC). To apply the EPC, the c.d.f. of the $CARL_0$ is required. For the case where only the μ_0 is estimated, we derived the c.d.f. of the $CARL_0$ for the first time in the literature. Using this result, we derive expressions that provide the values of m or adjustments to the limits to guarantee a desired performance in terms of the EPC.

Keywords: Parameter Estimation, $\bar{X}$ Chart Performance, Conditional Perspective, Unconditional Performance, False Alarm Rate

Resumo em Português

O gráfico $\bar{X}$ é uma das ferramentas mais usadas para monitorar a média de um processo. Para construir esse gráfico, a média (μ_0) ou o desvio padrão (σ_0) do processo devem ser estimados a partir de *m* amostras históricas. Nesse caso, o desempenho em controle do gráfico é afetado. Para estudar este efeito, recentemente surgiu a ideia de medir o desempenho dos gráficos de controle pela probabilidade de o número médio de amostras condicional ($CARL_0$) ser maior que um valor especificado pelo gestor. Isso é chamado de *Exceedance Probability Criterion* (EPC). Para aplicar o EPC, a c.d.f. do $CARL_0$ é necessária. Para o caso em que apenas a μ_0 é estimada, neste trabalho, a c.d.f. do $CARL_0$ foi derivada pela primeira vez na literatura. Usando esse resultado, foram derivadas também expressões que fornecem os valores de *m* ou ajustes nos limites para garantir um desempenho desejado em termos do EPC.

Palavras-Chaves: Desempenho do Gráfico $\bar{X}$, Desempenho Condicional, Desempenho Incondicional, Taxa de alarmes, Estimação de Parâmetros

1. Introduction

The effect of parameter estimation on the performance of the $\bar{X}$ control chart, since Shewhart (1939, p. 76), has being frequently studied in the literature of the Statistical Process Control (SPC) area. The in-control (IC) process mean (μ_0) and standard deviation (σ_0) are important parameters to calculate the control limits of the $\bar{X}$ chart. Usually these parameters are unknown and must be estimated from m reference samples each of size n when process is in control. This is called the Phase I in SPC area. For a literature review of SPC Phase I, the reader is referred to Chakraborti et al. (2009) and Jones-Farmer et al. (2014). With the estimated mean (μ_0) and standard deviation (σ_0), the $\bar{X}$ chart's control limits can be calculated using these estimates in the prospective process monitoring (called Phase II), where samples (also of size n) are collected at regular intervals. These samples are used to calculate the plotting statistic [i.e., the sample mean ($\bar{X}$)] which is compared with the control limits. If the plotting statistic is outside the control limits range, the probability that the process mean has changed must be high and the manager must find and correct a possible problem with the process.

Since estimators are random variables, when the control limits are estimated and used in Phase II, the performance of the control chart may be very different from the nominally advertised performance when parameters are known (and not estimated). This problem has been pointed out and analyzed by several authors. For reviews of the works on the effect of parameters estimation on the performance of control charts in general, see Jensen et al. (2006) and Psarakis et al. (2014).

Many researchers studying the effect of parameters estimation on the performance of $\bar{X}$ control chart have focused on the marginal distribution of the in-control (IC) run length (RL_0 – the number of Phase II samples until a false alarm), especially on its expected value, the so-called unconditional in-control average run length, denoted ARL_0 [see for example, Quesenberry (1993), Chen (1997), Chakraborti (2000 and 2006) and Goedhart et al., (2016)]. The ARL_0 is an overall average of all the infinite possible control charts each one construct with a possible realization of the estimators. Note that a specific user will have an estimate value different from that of another user. Thus, more recently, a great number of researchers advocated against the use of ARL_0 as a chart performance measure, because this overall expectation does not account for what is known as *the practitioner-to-practitioner variability* [see Saleh et al. (2015a,b)].

Recognizing this shortcoming, recently, some authors recommended to measure control charts performance and design them focusing on the probability of the *in-control average run length conditioned on the estimated parameters* (denoted $CARL_0$) be greater than a specified value (close to the nominal desired one). This is known as the Exceedance Probability Criterion (EPC) proposed by Albers and Kallenberg (2005) and Albers et al. (2005). Note that to calculate the EPC, the cumulative distribution function (c.d.f.) – or its reciprocal, the survival function – of the $CARL_0$ is required. For the $\bar{X}$ chart and the case where both the in-control process mean (μ_0) and the in-control process standard deviation (σ_0) are unknow and estimated (this is denoted Case UU), the exact c.d.f. expression of the $CARL_0$ was derived by Jardim et al. (2018a), while other authors relied on approximations for this distribution [see Goedhart et al. (2017 and 2018)]. For the case where μ_0 is considered known and the σ_0 is unknown and estimated (this is denoted Case KU), Jardim et al. (2018a,b) also derived the exact c.d.f. expression of the $CARL_0$. However, for the case where only the μ_0 is estimated and the σ_0 is specified (this is denoted Case UK: Unknown mean and Known standard deviation), the c.d.f. of the $CARL_0$ is unavailable in the literature.

For Cases UU and KU, designing the chart considering the *practitioner-to-practitioner variability* was a subject of much debate in the recent literature, see for example, Saleh et al (2015a,b), Faraz et al. (2017), Goedhart et al (2017 and 2018) and Jardim et al. (2018a,b). Some of these works are summarized next.

Saleh et al. (2015a,b) focused on the $\bar{X}$, X and the EWMA charts, and showed that using the standard deviation of the $CARL_0$, in addition to the average (ARL_0), as a performance measure, better accounts for practitioner-to-practitioner variability. To this end, they show that an impractical number of Phase I samples are required to guarantee that the standard deviation of the

$CARL_0$ is within at most 10% of the desired nominal ARL_0 value. Thus, they examined adjusting the control limits under the EPC using the bootstrap approach of Gandy and Kvaloy (2013). Goedhart et al. (2017 and 2018) also studied correcting the limits of the X and the $\bar{X}$ chart, respectively, based on the EPC considering approximations to the distribution of $CARL_0$ while Jardim et al. (2018a) based on the exact distribution of $CARL_0$. Faraz et al. (2017) also proposed a method to adjust the $\bar{X}$ chart in Case UU, however, their adjustment was not based in the EPC, instead, it was based on the *equal-tailed tolerance interval* together with the *Bonferroni Inequality* [see Krishnamoorthy and Mathew (2009, p. 4 and p.10)], which generates wider adjusted control limits if compared to the adjusted limits derived under the *EPC*.

With this background as motivation, in this paper, we provide an exact method and also derive an accurate approximate formula to obtain the c.d.f. of the $CARL_0$ distribution of the $\bar{X}$ chart in Case UK. Using this c.d.f., we provide expressions for (and calculate the) exact lower quantiles of the $CARL_0$ distribution which can be seen as prediction bounds for $CARL_0$. These results show that when the number of reference samples (m) is small, a lower quantile of the $CARL_0$ (such as the 0.1 and 0.05 quantiles) are much smaller than then the typically desired (nominal) value. This implies that the number of the reference samples plays a crucial role in the in-control performance of the chart when only the in-control process mean is estimated. Under this motivation and using the c.d.f. derived here, we then determine the exact number of Phase I samples needed to guarantee a desired in-control performance in terms of the EPC. A simple approximate formula which provides similar and accurate results is also derived here. The numbers of initial (Phase I) samples required that we obtained according to this criterion (such as 120 or 55 samples for example) are larger than the 25 or 30 samples recommended in most manual and text books in SPC [see, Montgomery (2009)]. Since such large amount of Phase I data may not be easily available for the users, next, as a practical solution, we derive exact and approximate adjustments to the control limits for a given number of Phase I samples at hand so that a desired conditional in-control performance is guaranteed in the spirit of the EPC.

Finally, it should be emphasized here that the present paper is part of a larger work in which we also study the other cases of parameters estimation (UU and KU) and also study the effect on the out-of-control performance of the $\bar{X}$ chart after the adjustments to the limits under the EPC.

This paper is organized as follows: In Section 2, we consider the $\bar{X}$ control chart limits and provide the exact expressions for the $CARL_0$. An exact method and an approximate expression for the c.d.f. of the $CARL_0$ are provided in Section 3. Prediction bounds for the $CARL_0$ is in Section 4. In Sections 5 and 6 we consider the minimum number of reference samples and the adjustment to the control limits in order to achieve a specified in-control performance under the EPC. Finally, conclusions are presented in Section 7.

2. The $\bar{X}$ Control Chart Model When the Process Mean is Estimated

In the present work, the observations of the process quality variable (X) are considered i.i.d. and normally distributed, like is traditionally done for the $\bar{X}$ Control Chart. In the ideal case, the in-control process mean (μ_0) and standard deviation (σ_0) are both known or specified. In this situation, the upper and lower control limits (UCL and LCL) of the L-sigma $\bar{X}$ Control Chart with subgroups of size n are given, respectively by

$$UCL = \mu_0 + L\frac{\sigma_0}{\sqrt{n}} \tag{1}$$

and

$$LCL = \mu_0 - L\frac{\sigma_0}{\sqrt{n}}, \tag{2}$$

where the control limit factor L is either a value such as 3 (the widely used "3 sigma limits") or is chosen so as to provide a nominal in-control average run length ($arl_{0,nom}$) or a false-alarm rate ($\alpha = 1/arl_{0,nom}$). In the latter case, we have $L = z_{\alpha/2} = \Phi^{-1}(1-\alpha/2)$, where $\Phi(\cdot)$ denotes the standard normal c.d.f. Thus, the usual 3-sigma limits correspond to a nominal false alarm rate of $\alpha = 0.0027$ and a nominal in-control average run length of $arl_{0,nom} = 370.4$. In the present work, we are concerned only with the case that μ_0 is unknown and need to be estimated from a Phase I data, consisting of m subgroups of size n, taken from the process when it is in control. The most common estimator for the μ_0 is the $\bar{\bar{X}}$, the grand mean of the m Phase I samples:

$$\bar{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m} \bar{X}_i, \tag{3}$$

where $\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$ and X_{ij} denotes the j-th observation of the i-th Phase I sample.

A signal occurs when, in a given Phase II sample, the average $\bar{X}$ (i.e., the plotting statistic) lies outside the control limits. Given the expressions for the control limits (Eq. 1 and Eq. 2) and replacing μ_0 by $\bar{\bar{X}}$, the conditional probability of a signal (CPS) in case UK for any Phase II sample can be written as

$$CPS = P\left(Signal|\bar{\bar{X}}\right) = 1 - P\left(\bar{\bar{X}} - L\frac{\sigma_0}{\sqrt{n}} \le \bar{X} \le \bar{\bar{X}} + L\frac{\sigma_0}{\sqrt{n}}\right). \tag{4}$$

It is then evident from Eq. (4) that the probability of a signal in Phase II is conditioned on the value of the estimator $\bar{\bar{X}}$. If in Phase II $\bar{X} \sim N(\mu_0, \sigma_0^2/n)$, the process is considered in control (IC). In this case, and considering that $Z = \left(\frac{\bar{\bar{X}}-\mu_0}{\sigma_0}\right)\sqrt{mn}$ follows a standard normal distribution (see Chakraborti, 2000), the Conditional False Alarm Rate, $CFAR$, can be expressed by:

$$CFAR = P(Signal|Z, \text{IC}) = 1 - \left[\Phi\left(\frac{Z}{\sqrt{m}} + L\right) - \Phi\left(\frac{Z}{\sqrt{m}} - L\right)\right]. \tag{5}$$

It is interesting to note that $CFAR$ is not a function of n. Thus, the in-control performance of the $\bar{X}$ chart in Case UK does not depend on the size of each Phase I samples. Given that the in-control conditional run length (RL_0) distribution of the $\bar{X}$ chart is geometric with parameter $CFAR$ [see, for example, Chakraborti (2000)], then its expected value, the conditional in-control average run length $CARL_0$, is:

$$CARL_0 = \frac{1}{CFAR} = \left(1 - \left[\Phi\left(\frac{Z}{\sqrt{m}} + L\right) - \Phi\left(\frac{Z}{\sqrt{m}} - L\right)\right]\right)^{-1}. \tag{6}$$

To visualize the effect of the number of Phase I samples (m), Figure 1 presents the $CARL_0$ curves parametrized by m in function Z for $m = 10,\ 20, 50, 100, 500$ and $\alpha = 0.0027$ (i.e., a nominal ARL_0 of 307.4 and $L = 3$).

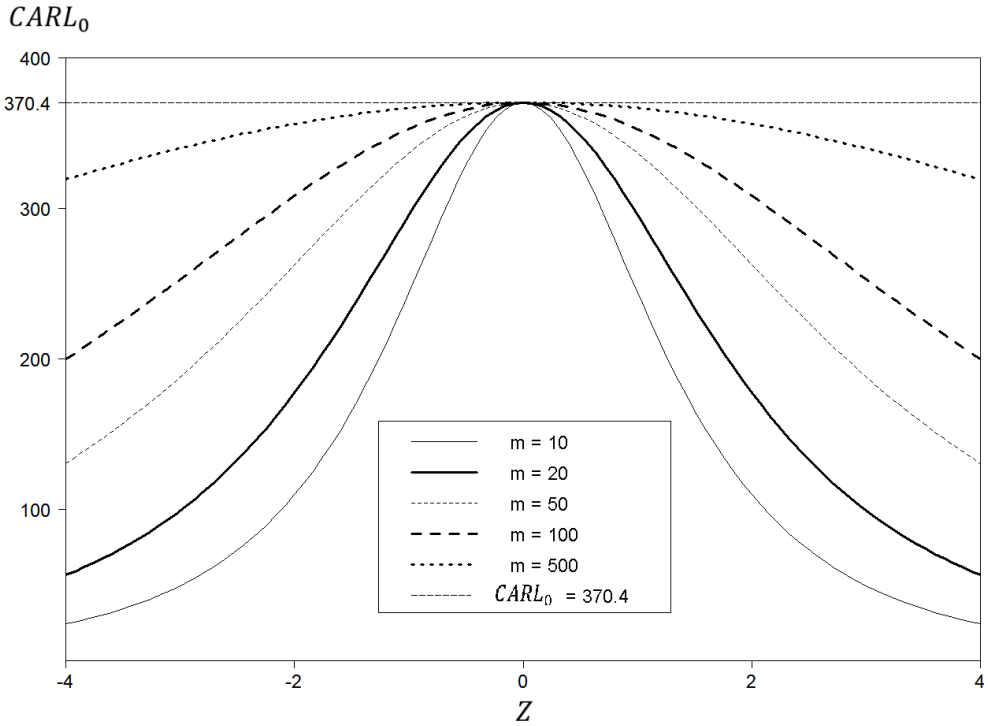


Figure 1. $CARL_0$ as function of Z for $m = 10, 20, 50, 100, 500$ and $arl_{0,nom} = 370.4$ (i.e., $L = 3$).

Figure 1 clearly shows the effect of the number of Phase I samples (m) on the performance of the $\bar{X}$ control chart when the in-control process mean is estimated. The horizontal line corresponds to the value of ARL_0 in the known parameters case (Case KK) and can be considered the target when the 3-Sigma limits are used. So, it can be seen, that the curves of $CARL_0$ are significantly closer to the horizontal line (the target) when m is large (compare for example the curves for $m = 10$ and for $m = 500$). This means that the difference between the target and the $CARL_0$ values is considerably more likely to be larger when m is small.

Another interesting behavior of the $CARL_0$ can be verified in Figure 1: the maximum possible value of $CARL_0$ is the exact nominal desired value in case KK (i.e., 370.4, for $L = 3$). The fact that $CARL_0$ cannot be larger than the its nominal value is a particular and remarkable property of case UK. This does not happen in the other parameter estimation cases (UU and KU), see for example Jardim et al. (2018a).

Figures 1 provides a good insight of the conditional in-control average run length ($CARL_0$) behavior when the process mean is estimated (such as the maximum possible value of $CARL_0$ or "how distant" is its curve from the nominal "target" line). However, they do not clear provide the distribution of this random variable. In the next section, the distributions of the $CARL_0$ are derived together with an approximation formula for it. This distribution clearly shows the effect of the in-control process mean estimation on the $\bar{X}$ chart performance and it is also required to calculate the results for the Exceedance Probability Criterion (EPC), as noted in the Introduction.

3. The cumulative distribution function of $CARL_0$

According to Equation (6), the $CARL_0$ is a function of only the random variable Z and it is not a monotonic function. Because of this, it is not possible to derive an exact closed-form expression for the c.d.f. of $CARL_0$ (F_{CARL_0}). Given this, in this section, we present the equations to calculate exact values of F_{CARL_0} with a search algorithm and also a formula to calculate approximate values of F_{CARL_0}. A search algorithm, such as the secant method, provides extremely

accurate results, since the user can specify the desired accuracy in order to find a result that is exact up to a specified number of significant digits. For more details of the search algorithm, see Jardim et al. (2018a). For now, the c.d.f. of $CARL_0$ (F_{CARL_0}) can be expressed as:

$$F_{CARL_0}(w) = P(CARL_0 \leq w) = \begin{cases} 0, & w \leq 1 \\ 1 - (\Phi(z_2) - \Phi(z_1)) & 1 < w < 370.4 \\ 1, & w \geq 370.4 \end{cases} \quad (7)$$

where z_1 and z_2 are the only two solutions, according to Equation (6), of

$$\left[1 - \left[\Phi\left(\frac{Z}{\sqrt{m}} + L\right) - \Phi\left(\frac{Z}{\sqrt{m}} - L\right)\right]\right]^{-1} = w, \quad (8)$$

for Z, being $z_1 < 0 < z_2$. Note again that z_1 and z_2 can be precisely found using a search algorithm as the secant method.

Equation (7) can be explained as follows: there are only two solutions for Z of $CARL_0 = w$ (see also figure 1) because $CARL_0$ as a function of Z is increasing on $(-\infty,\ 0]$ and decreasing on $[0, \infty)$ so that $CARL_0$ varies in the interval $(1, max(CARL_0) = 370.4]$. The value of $CARL_0$ tends to one when Z tends to $-\infty$ or ∞. Thus, the probability that $CARL_0 < w$ (i.e., the c.d.f. of $CARL_0$) is the probability that Z is smaller than z_1 or larger than z_2.

Note that Equations (7) and (8) provide exact values of the c.d.f. of $CARL_0$ (F_{CARL_0}). However, to calculate these equation, a software (such as RStudios) will be required to run an search algorithm to find z_1 and z_2. However, a simple approximate formula for F_{CARL_0} can be derived starting from Eq. (6) where much less resources are needed. This is done next. From Eq. (6), one can write

$$\begin{aligned} F_{CARL_0}(w) = P(CARL_0 \leq w) &= P\left(\left[1 - \left[\Phi\left(\frac{Z}{\sqrt{m}} + L\right) - \Phi\left(\frac{Z}{\sqrt{m}} - L\right)\right]\right]^{-1} \leq w\right) \\ &= P\left(P\left(\frac{Z}{\sqrt{m}} - L \leq Z_1 \leq \frac{Z}{\sqrt{m}} + L\right) \leq 1 - \frac{1}{w}\right), \end{aligned} \quad (9)$$

where Z_1 follows a standard normal random variable, so Eq. (9) can be expressed as

$$\begin{aligned} F_{CARL_0}(w) = P(CARL_0 \leq w) &= P\left(P\left(-L \leq Z_1 - \frac{Z}{\sqrt{m}} \leq +L\right) \leq 1 - \frac{1}{w}\right) \\ &= P\left(P\left(\left(Z_1 - \frac{Z}{\sqrt{m}}\right)^2 \leq L^2\right) \leq 1 - \frac{1}{w}\right), \end{aligned} \quad (10)$$

where $\left(Z_1 - \frac{Z}{\sqrt{m}}\right)^2$ follows a non-central chi-squared distribution with 1 degree of freedom (d.f.) and non- centrality parameter $\frac{Z^2}{m}$. So, defining $\left(Z_1 - \frac{Z}{\sqrt{m}}\right)^2 = \chi^2_{1,\left[\frac{Z^2}{m}\right]}$, one can write:

$$F_{CARL_0}(w) = P\left(F_{CARL_0} \leq w\right) = P\left(P\left(\chi^2_{1,\left[\frac{Z^2}{m}\right]} \leq L^2\right) \leq 1 - \frac{1}{w}\right). \quad (11)$$

Using the approximation result derived by Cox and Reid (1987) for a non-central chi-squared distribution where $P\left(\chi^2_{1,\left[\frac{Z^2}{m}\right]} \leq L^2\right) \approx P\left(\chi^2_1 \leq \frac{L^2}{1+\frac{Z^2}{m}}\right)$ with χ^2_1 being a central chi-squared random variable with 1 d.f., Eq. (11) can be written as

$$F_{CARL_0}(w) = P\left(F_{CARL_0} \leq w\right) \approx P\left(P\left(\chi^2_1 \leq \frac{L^2}{1+\frac{Z^2}{m}}\right) \leq 1 - \frac{1}{w}\right)$$
$$= 1 - P\left(Z^2 \leq m\left(\frac{L^2}{F^{-1}_{\chi^2_1}\left(1-\frac{1}{w}\right)} - 1\right)\right), \quad (12)$$

where $F^{-1}_{\chi^2_1}\left(1-\frac{1}{w}\right)$ denotes the $\left(1-\frac{1}{w}\right)$-quantile of a chi-squared distribution with 1 degree of freedom (d.f.). Since, Z^2 also follows a chi-squared distribution with 1 d.f., finally, the final approximate formula for F_{CARL_0} can be expressed as

$$F_{CARL_0}(w) = P\left(F_{CARL_0} \leq w\right) \approx 1 - F_{\chi^2_1}\left(m\left(\frac{L^2}{F^{-1}_{\chi^2_1}\left(1-\frac{1}{w}\right)} - 1\right)\right) \quad (13)$$

The c.d.f.'s of the $CARL_0$ for $m = \{10,20,50,200,500\}$ and $\alpha = 0.0027$ (i.e., $L = 3$) are shown in Figure 2. Exact values are calculated according to Equations (7) and (8), in black, and the approximate values are calculated according to Equation (13) in gray. Note that the approximate formula provides accurate results.

From Figure 2, it is evident the impact of m on the $CARL_0$ distributions. When m gets larger (such as $m = 500$), the c.d.f. curves are much "closer" to the vertical line ("the target") of the nominal value of 370.4.

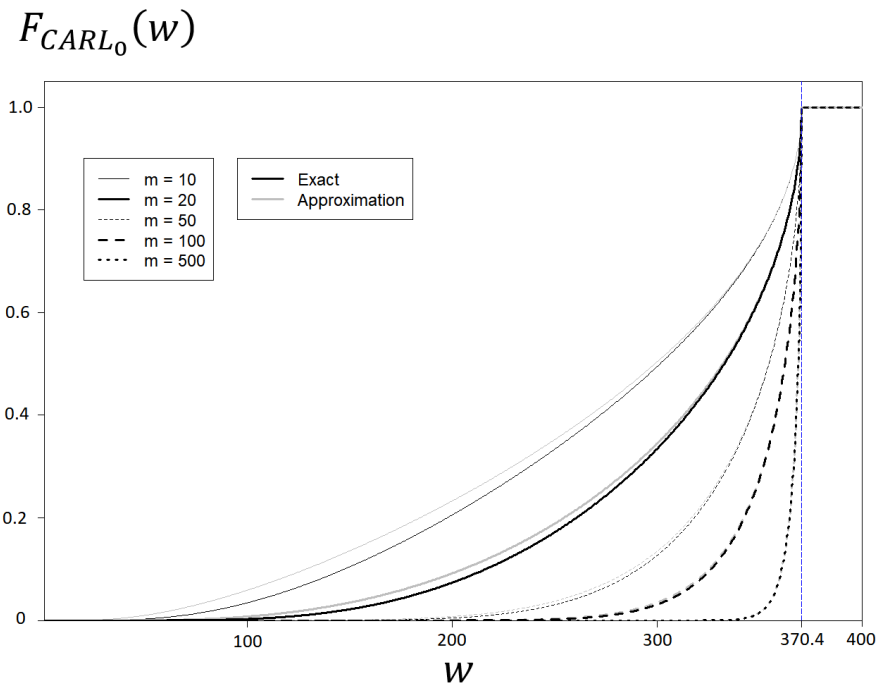


Figure 1. c.d.f. of $CARL_0$ for $m = 10, 20, 50, 100, 500$ and $\alpha = 0.0027$ (i.e., $L = 3$).

4. Prediction Bounds for $CARL_0$

Since $CARL_0$ is a random variable, we next consider the problem of predicting it. For example, it may be of interest to know, in practice, in a given Phase II application, how far $CARL_0$ can be from the nominal desired value. Put in another way, for a given m, it may be of practical interest to know the value of $CARL_0$ (denote here as arl_p) that has a specified high probability $1 - p$ (say, 90%) of being exceeded. This means that we need to find the value arl_p such that when the process is in control:

$$P\left(CARL_0 \geq arl_p\right) = 1 - p. \tag{14}$$

This means that the required arl_p is the p-quantile of $CARL_0$. One can use Equations (7) and (8) to find arl_p by numerically solving the following system of equations:

$$\begin{cases} \Phi(z_2) - \Phi(z_1) = 1 - p \\ \Phi\left(\frac{z_i}{\sqrt{m}} + L\right) - \Phi\left(\frac{z_i}{\sqrt{m}} - L\right) = 1 - \frac{1}{arl_p}, \quad i = 1,2 \end{cases} \tag{15}$$

for z_1, z_2 and arl_p. Note again that z_1, z_2 and arl_p can be found with a search algorithm, such as the secant method. From Eq. (13), it is also possible to derive a simple approximate formula for arl_p by replacing w by arl_p and rearranging the terms, as shown below.

$$1 - F_{\chi_1^2}\left(1 - F_{\chi_1^2}\left(m\left(\frac{L^2}{F_{\chi_1^2}^{-1}\left(1-\frac{1}{arl_p}\right)} - 1\right)\right)\right) = p \quad \therefore \quad arl_p \approx \left[1 - F_{\chi_1^2}\left(\frac{L^2}{\frac{F_{\chi_1^2}^{-1}(1-p)}{m}+1}\right)\right]^{-1} \tag{16}$$

Table 1 presents the values of arl_p for $p = 5\%, 10\%, 15\%, 20\%$ for some values of m and $L = 3$. The exact arl_p values were calculated numerically using the system of equations (15) with a search method, the values obtained using the simple approximation given by Eq. (16) is in bold. As it can be seen, when m is small, the arl_p values are much smaller than the desired 370.4. For example, for $m = 10$ and $p = 5\%$, $arl_p = 113.7$, less than the half of 370.4. Note also, that the approximation given by Eq. (16) works reasonably well for $m \geq 50$.

The results from Table 1 show that for a small number of Phase I samples, such as $m = 25$, if the 3-Sigma limits are used, the variability of $CARL_0$ is large, meaning that the realization of this random variable may be very different from 370.4, often much smaller. This implies that the number of the reference samples plays a crucial role in the in-control performance of the chart when only the in-control process mean is estimated and the 3-sigma limits is used ($L = 3$). Given this, in next section, we calculate the exact minimum numbers of Phase I samples that guarantees that $CARL_0$ is larger than a specified value (close to the nominal) with larger probability (in terms of the *EPC*). An approximate formula to find this minimum number of Phase I samples is also derived.

Table 1 - arl_p values of $CARL_0$ for $L = 3$.

	m													
	5		**10**		**25**		**50**		**100**		**500**		**1000**	
p	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**
0.05	59.1	**41.5**	113.7	**92.8**	204.1	**191.5**	265.9	**260.4**	310.5	**308.6**	356.8	**356.7**	363.5	**363.5**
0.1	84.3	**63.8**	147.3	**128.5**	237.1	**228.6**	290.8	**287.5**	326.3	**325.3**	360.8	**360.7**	365.5	**365.5**
0.15	106.8	**85.8**	174.1	**158.1**	259.9	**253.8**	306.5	**304.4**	335.8	**335.1**	363.0	**362.9**	366.7	**366.6**
0.2	128.4	**108.2**	197.5	**184.2**	277.6	**273.2**	318.1	**316.6**	342.5	**342.0**	364.5	**364.5**	367.4	**367.4**

5. Finding m for a Guaranteed In-Control Performance

As highlighted in the previous section, another relevant question for the practitioner may be the minimum number m of Phase I reference samples that guarantees, with a specified high probability $1-p$ (say, 0.9), that $CARL_0$ is larger than a value which may be slightly smaller than the nominal ($arl_{0,nom}$). This is the Exceedance Probability Criterion (EPC) and can be formulated as follows: Given the values of $arl_{0,nom}$, ε and p find the minimum number of in-control Phase I samples, m, such that

$$P\left(CARL_0 \geq (1-\varepsilon)arl_{0,nom}\right) = 1-p \tag{17}$$

This problem is like the one in Section 4, with the difference that now arl_p is given and is equal to $(1-\varepsilon)arl_{0,nom}$ and m is the unknown that needs to be found. Note that, since m is an integer, a perfect match is generally not possible, so, re-stating the problem, m should be the smallest integer such that $P\left(CARL_0 \geq (1-\varepsilon)arl_{0,nom}\right) \geq 1-p$ is true. The quantity ε is called the *tolerance factor* which allows flexibility in choosing the lower bound to the $CARL_0$ one would like to assure. This means that users can actually specify a $CARL_0$ value they will be willing to tolerate, that is at least $100\varepsilon\%$ lower than the nominal $arl_{0,nom}$ with a high probability (corresponding to a small specified p).

For case UK, m can be exactly found using a simple search algorithm (as the Secant Method, for example) since F_{CARL_0} is a monotonic increasing function of m. Basically, this means that we need to solve the following system of equations:

$$\begin{cases} \Phi(z_2) - \Phi(z_1) = 1-p \\ \Phi\left(\frac{z_i}{\sqrt{m}} + L\right) - \Phi\left(\frac{z_i}{\sqrt{m}} - L\right) = 1 - \frac{1}{(1-\varepsilon)arl_{0,nom}}, \quad i = 1,2 \end{cases} \tag{18}$$

for m, z_1 and z_2. Again, using Eq. (13), replacing w by $(1-\varepsilon)arl_{0,nom}$ and rearranging the terms it is also possible to derive a simple approximate formula for the smallest integer m as shown below:

$$1 - F_{\chi_1^2}\left(1 - F_{\chi_1^2}\left(m\left(\frac{L^2}{F_{\chi_1^2}^{-1}\left(1-\frac{1}{(1-\varepsilon)arl_{0,nom}}\right)} - 1\right)\right)\right) = p \quad \therefore \quad m \approx \left\lceil \frac{F_{\chi_1^2}^{-1}(1-p)}{\frac{L^2}{F_{\chi_1^2}^{-1}\left(1-\frac{1}{(1-\varepsilon)arl_{0,nom}}\right)} - 1} \right\rceil \tag{19}$$

where $\lceil a \rceil$ denotes the smallest integer greater or equal to a.

Table 2 shows the minimum number of in-control Phase I samples, m. We considered $L = 3$ ($arl_{0,nom} = 370.4$), $p = 5\%, 10\%, 15\%, 20\%$ and $\varepsilon = 10\%, 20\%, 30\%, 40\%, 50\%$. As can be seen in Table 2, for small values of ε and p, one needs a large number of Phase I samples to guarantee an in-control performance in terms of the EPC. For example, for $p = 10\%$ and $\varepsilon = 10\%$, an exact 120 samples are required. These values are larger than the 30 and 25 samples recommended by most manuals and textbooks [see Montgomery (2009)]. It is important to highlight here that the size of each sample (n) does not affect the in-control performance [note that Equations (18) and (19) do not depend on n].

In case the required m is infeasible and relaxing the value of either ε or p or both is unacceptable on practical grounds, a possible solution is to change the value of the control limit factor L (instead of using $L = 3$ – the most common 3-sigma limits), given a fixed value of m, at hand, in order to satisfy the EPC. This is discussed in the next section.

Table 2. Minimum number of in control Phase I samples, m, required for or $P(CARL_0 \geq (1-\varepsilon)370.4) = 1-p$ (i.e., $L = 3$ and $arl_{0,nom} = 370.4$).

	ε									
	0.1		**0.2**		**0.3**		**0.4**		**0.5**	
p	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**
0.05	171	**175**	78	**81**	46	**50**	31	**34**	21	**24**
0.1	120	**123**	55	**57**	33	**35**	22	**24**	15	**17**
0.15	92	**94**	42	**44**	25	**27**	17	**18**	12	**13**
0.2	73	**75**	33	**35**	20	**22**	13	**15**	9	**11**

6. Adjustment of the Limits for a Guaranteed In-Control Performance

In the previous chapter, we saw that the minimum numbers of reference samples required to guarantee some conditional in-control performances can be very large and may be infeasible in many practical situations. Given this possible problem, in this section, we present exact and approximate adjustments to the control limits of the $\bar{X}$ chart for given values of m that guarantee an in-control performance in terms of the EPC. So basically, the idea is to replace the limit factor L in Equations (1) and (2) by L^*, where L^* represents the value of the control limit factor that guarantees that

$$P\left(CARL_0 \geq (1-\varepsilon)arl_{0,nom}\right) = 1-p \tag{20}$$

for a given value of $arl_{0,nom}$, ε, p and m. Note that Eq. (20) is equal to Eq. (17), in the sense that both represent the in-control performance goal under the EPC. We repeated Eq. (17) here for didactic reason. Since there is no closed-form solution for the exact expression of the c.d.f. of $CARL_0$, one can use Equations (7) and (8) to find L^* by solving the following system of equations:

$$\begin{cases} \Phi(z_2) - \Phi(z_1) = 1-p \\ \left[\Phi\left(\frac{z_i}{\sqrt{m}} + L^*\right) - \Phi\left(\frac{z_i}{\sqrt{m}} - L^*\right)\right] = 1 - \frac{1}{(1-\varepsilon)arl_{0,nom}}, \qquad i = 1,2 \end{cases} \tag{21}$$

for L^*, z_1 and z_2. This can be done numerically with a search algorithm. Using the approximate formula for the c.d.f. given by Eq. (13), one can also derive the following approximate formula for L^*:

$$L^* \approx \sqrt{\left(\frac{F_{\chi^2_1}^{-1}(1-p)}{m} + 1\right) F_{\chi^2_1}^{-1}\left(1 - \frac{1}{(1-\varepsilon)arl_{0,nom}}\right)} \tag{22}$$

Table 3 shows the exact [using Equations (21)] and approximate [in bold, using equation (22)] values of L^* for some values of m, for $\varepsilon = 0, 0.05, 0.1, 0.15, 0.2$, $p = 0.05$,0.1, 0,15,0.2 and $arl_{0,nom} = 370.4$. As can be seen in Table 3, the approximate formula (22) provides accurate results for $m \geq 25$.

According to Table 3, for $m = 25$ (the most recommended number of Phase 1 samples in the literature), to guarantee an in-control performance in terms of EPC for $p = 10\%$ and $\varepsilon = 0$, the user must replace $L = 3$ (the 3-Sigma Limits) for $L^* = 3.19$ in Equations (1) and (2). Note that the adjustments based on the EPC of the $\bar{X}$ control chart in most of the parameters combinations (p, ε and m) make the limits wider than the 3-sigma limits (note that often $L^* > 3$). Also note that since p and ε values can be balanced in order to achieve some desired specific in-control performance, the EPC formulation allows the user some flexibility during the implementation of the control chart.

Table 3. Values of L^* to achieve $P(CARL_0 \geq (1-\varepsilon)370.4) = 1-p$ for $\varepsilon = 0, 0.05, 0.1, 0.15, 0.2$, $p = 0.05, 0.1, 0,15, 0.2$ and $m = 25, 75, 50, 100, 150, 200, 250, 500, 1000$.

		ε											
		0		**0.1**		**0.2**		**0.3**		**0.4**		**0.5**	
m	p	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**	Exact	**Approx**
m = 5	**0.05**	3.66	**3.99**	3.62	**3.95**	3.59	**3.90**	3.54	**3.84**	3.49	**3.78**	3.43	**3.70**
	0.1	3.52	**3.72**	3.48	**3.68**	3.45	**3.64**	3.40	**3.59**	3.35	**3.53**	3.29	**3.45**
	0.15	3.43	**3.57**	3.39	**3.53**	3.36	**3.49**	3.31	**3.44**	3.26	**3.38**	3.20	**3.31**
	0.2	3.36	**3.46**	3.33	**3.42**	3.29	**3.38**	3.24	**3.33**	3.19	**3.27**	3.13	**3.21**
m = 10	**0.05**	3.41	**3.53**	3.37	**3.49**	3.33	**3.45**	3.29	**3.40**	3.24	**3.34**	3.17	**3.27**
	0.1	3.31	**3.38**	3.28	**3.34**	3.24	**3.30**	3.19	**3.26**	3.14	**3.20**	3.08	**3.14**
	0.15	3.25	**3.30**	3.22	**3.26**	3.18	**3.22**	3.13	**3.17**	3.08	**3.12**	3.02	**3.06**
	0.2	3.21	**3.24**	3.17	**3.20**	3.13	**3.16**	3.09	**3.12**	3.04	**3.06**	2.98	**3.00**
m = 25	**0.05**	3.19	**3.22**	3.16	**3.19**	3.12	**3.15**	3.08	**3.10**	3.03	**3.05**	2.97	**2.99**
	0.1	3.14	**3.16**	3.11	**3.12**	3.07	**3.09**	3.03	**3.04**	2.98	**2.99**	2.92	**2.93**
	0.15	3.11	**3.12**	3.08	**3.09**	3.04	**3.05**	3.00	**3.01**	2.95	**2.96**	2.89	**2.89**
	0.2	3.09	**3.10**	3.06	**3.06**	3.02	**3.03**	2.98	**2.98**	2.93	**2.93**	2.87	**2.87**
m = 50	**0.05**	3.11	**3.11**	3.07	**3.08**	3.03	**3.04**	2.99	**3.00**	2.94	**2.95**	2.88	**2.89**
	0.1	3.08	**3.08**	3.04	**3.05**	3.01	**3.01**	2.96	**2.97**	2.91	**2.92**	2.85	**2.86**
	0.15	3.06	**3.06**	3.03	**3.03**	2.99	**2.99**	2.95	**2.95**	2.90	**2.90**	2.84	**2.84**
	0.2	3.05	**3.05**	3.01	**3.02**	2.98	**2.98**	2.94	**2.94**	2.89	**2.89**	2.83	**2.83**
m = 100	**0.05**	3.05	**3.06**	3.02	**3.02**	2.98	**2.99**	2.94	**2.94**	2.89	**2.89**	2.83	**2.83**
	0.1	3.04	**3.04**	3.01	**3.01**	2.97	**2.97**	2.93	**2.93**	2.88	**2.88**	2.82	**2.82**
	0.15	3.03	**3.03**	3.00	**3.00**	2.96	**2.96**	2.92	**2.92**	2.87	**2.87**	2.81	**2.81**
	0.2	3.02	**3.02**	2.99	**2.99**	2.95	**2.96**	2.91	**2.91**	2.86	**2.86**	2.80	**2.80**
m = 500	**0.05**	3.01	**3.01**	2.98	**2.98**	2.94	**2.94**	2.90	**2.90**	2.85	**2.85**	2.79	**2.79**
	0.1	3.01	**3.01**	2.98	**2.98**	2.94	**2.94**	2.90	**2.90**	2.85	**2.85**	2.79	**2.79**
	0.15	3.01	**3.01**	2.97	**2.97**	2.94	**2.94**	2.90	**2.90**	2.85	**2.85**	2.79	**2.79**
	0.2	3.00	**3.00**	2.97	**2.97**	2.94	**2.94**	2.89	**2.89**	2.85	**2.85**	2.79	**2.79**
m = 1000	**0.05**	3.01	**3.01**	2.97	**2.97**	2.94	**2.94**	2.89	**2.89**	2.85	**2.85**	2.79	**2.79**
	0.1	3.00	**3.00**	2.97	**2.97**	2.93	**2.93**	2.89	**2.89**	2.84	**2.84**	2.79	**2.79**
	0.15	3.00	**3.00**	2.97	**2.97**	2.93	**2.93**	2.89	**2.89**	2.84	**2.84**	2.78	**2.78**
	0.2	3.00	**3.00**	2.97	**2.97**	2.93	**2.93**	2.89	**2.89**	2.84	**2.84**	2.78	**2.78**

7. Conclusions

In this paper, the impact of the process mean (μ_0) estimation on the in-control performance of the $\bar{X}$ control chart was analysed. This study is new in the literature because almost all the previous works that studied the performance of the $\bar{X}$ chart when parameters are estimated focused on the case where only the in-control process standard deviation (σ_0) is estimated or the case where both μ_0 and σ_0 are estimated. This work also follows a new trend in the Statistical Process Control literature that advocates measuring charts performance by the probability of the conditional in-control average run length (denoted $CARL_0$) being greater than a given specified value (close or slightly smaller than the nominal value). This is called the Exceedance Probability Criterion (EPC).

To apply the EPC, we derived a method that provided exact values of the cumulative distribution function (c.d.f.) of the $CARL_0$. A simple and accurate approximate formula for the c.d.f. of $CARL_0$ was also derived here. Using these results, the exact lower quantiles of $CARL_0$ of the $\bar{X}$ chart, which constitute prediction bounds for $CARL_0$, were calculated and tabulated. These results show that when m is small, the lower quantiles of the $CARL_0$ (such as the 0.05 and 0.1 quantiles) are much smaller than the desired nominal ARL_0, meaning that the $CARL_0$ can be much smaller than expected when the $\bar{X}$ chart is designed with estimated process mean only.

In order to avoid unacceptably low $CARL_0$ values, expressions were derived, based on the c.d.f.´s derived here, in order to calculate exact and approximate values of m (the number of Phase I samples) required to guarantee a desired in-control performance in terms of the EPC. These results showed that depending on the practitioner's tolerances, m can be large, such as 50 samples. These numbers are larger than the ones recommended in most textbooks and manuals of SPC.

Given the large numbers (m) of Phase I samples required, also in this work, it was derived corrections to control limits so that some desired in-control performance in terms of the EPC is

achieved for a given m value. To this end, exact expression and approximate formulas for these corrections were derived and provided here. In most of the situations, these adjustments make the control limits wider (compared to the 3-sigma limits) which may affect the out-of-control performance of the chart, since wider control limits may delay the detection (signal) of real changes in the process mean. For future works, we recommend an out-of-control performance study after the adjustments proposed here for the $\bar{X}$ chart with estimated process mean in order to provide the best trade-off between the adjustments that guarantee an in-control performance and a not considerable deterioration of the out-of-control performance.

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