On the hull number for complementary prisms of connected graphs G and $\overline{G}$

Diane Castonguay
Instituto de Informática
Universidade Federal de Goiás, GO, Brasil
diane@inf.ufg.br

Erika M. M. Coelho
Instituto de Informática
Universidade Federal de Goiás, GO, Brasil
erikamorais@inf.ufg.br

Hebert Coelho
Instituto de Informática
Universidade Federal de Goiás, GO, Brasil
hebert@inf.ufg.br

Julliano R. Nascimento
Instituto de Informática
Universidade Federal de Goiás, GO, Brasil
jullianorn@inf.ufg.br

ABSTRACT

In the geodetic convexity, a set of vertices S of a graph G is *convex* if all vertices belonging to any shortest path between two vertices of S lie in S. The *convex hull* $H(S)$ of S is the smallest convex set containing S. If $H(S) = V(G)$, then S is a *hull set*. The cardinality $h(G)$ of a minimum hull set of G is the *hull number* of G. The *complementary prism* $G\overline{G}$ of a graph G arises from the disjoint union of the graph G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. Motivated by previous work, we study the hull number for complementary prisms when both G and $\overline{G}$ are connected. When G is a split graph, we show that the hull number is unlimited. Otherwise, when G is a non-split graph, it is limited by 3.

KEYWORDS. Geodetic convexity, hull number, complementary prisms.

Paper topics: Theory and Algorithms in Graphs (TAG).

https://proceedings.science/p/85314?lang=pt-br

1. Introduction

For a finite and simple graph G with vertex set $V(G)$, a *graph convexity* on $V(G)$ is a collection $\mathcal{C}$ of subsets of $V(G)$ such that $\emptyset, V(G) \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. The sets in $\mathcal{C}$ are called *convex sets* and the *convex hull* $H_{\mathcal{C}}(S)$ in $\mathcal{C}$ of a set S of vertices of G is the smallest set in $\mathcal{C}$ containing S. Some natural convexities in graphs are defined by a set $\mathcal{P}$ of paths in G, in a way that a set S of vertices of G is convex if and only if for every path $P = v_0v_1 \dots v_l$ in $\mathcal{P}$ such that v_0 and v_l belong to S, all vertices of P belong to S.

The definition of convex sets in graphs originally come from Euclidean geometry, in which a set S is convex if every line segment between two points of S remains in S. The concepts of convexity in graphs can be applied to model contexts envolving some disseminating processes between entities, e.g. marketing strategies [Domingos and Richardson, 2001], spread of disease and opinion [Dreyer and Roberts, 2009], and distributed computing [Peleg, 2002].

In this paper we study the convexity related to shortest paths in graphs, the *geodetic convexity* $\mathcal{C}$. Given a graph G, the *closed interval* $I[u, v]$ of a pair $u, v \in V(G)$ consists of all vertices lying in any shortest (u, v)-path in G. For a set $S \subseteq V(G)$, the *closed interval* $I[S]$ is the union of all sets $I[u, v]$ for $u, v \in S$. If $I[S] = S$, then S is a *convex set*. The *convex hull* $H_{\mathcal{C}}(S)$ of S is the smallest convex set containing S. Since a graph G uniquely determines its convexity $\mathcal{C}$, we may write $H(S)$, instead of $H_{\mathcal{C}}(S)$. If $H(S) = V(G)$ we say that S is a *hull set* of G. The cardinality $h(G)$ of a minimum hull set of G is called the *(geodetic) hull number* of G.

Everett and Seidman [1985] introduced the concept of hull number in the geodetic convexity. For some later results see, e.g. [Araujo et al., 2013; Dourado et al., 2010a; Hernando et al., 2005]. The computation of hull number is NP-hard for bipartite graphs [Araujo et al., 2013], partial cubes [Albenque and Knauer, 2014], and P_9-free graphs [Dourado et al., 2016], but it can be computed in polynomial time for cographs, split graphs [Dourado et al., 2009], $(q, q-4)$-graphs [Araujo et al., 2013], $\{paw, P_5\}$-free graphs [Dourado et al., 2016], and distance-hereditary graphs [Kanté and Nourine, 2016].

The hull number is also studied in other graph convexities. It can be determined in polynomial time for general graphs in *triangle path convexity* [Dourado and Sampaio, 2016], related to paths which allow only short chords, and *monophonic convexity* [Dourado et al., 2010b], related to induced paths. In *P_3-convexity*, that consider paths with three vertices, computing the hull number for general graphs is NP-complete [Centeno et al., 2011]. However, Duarte et al. [2015] showed that the hull number of complementary prisms can be determined efficiently, more specifically, it is limited by 5 when G and $\overline{G}$ are connected.

The *complementary prisms* were introduced by Haynes et al. [2007] as a variation of the well-known *prism* of a graph [Hammack et al., 2011]. For a graph G with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$, the *complementary prism* of G is the graph denoted by $G\overline{G}$ with vertex set $V(G\overline{G}) = \{v_1, \dots, v_n\} \cup \{\overline{v}_1, \dots, \overline{v}_n\}$ and edge set $E(G\overline{G}) = E(G) \cup \{\overline{v}_i\overline{v}_j : 1 \leq i < j \leq n$ and $v_iv_j \notin E(G)\} \cup \{v_1\overline{v}_1, \dots, v_n\overline{v}_n\}$. Let G be a graph and $\overline{G}$ its complement. For every vertex $v \in V(G)$ we denote $\overline{v} \in V(\overline{G})$ as its *corresponding vertex*, and for a set $X \subseteq V(G)$, we let $\overline{X}$ be the *corresponding set* of vertices in $V(\overline{G})$. Let $S \subseteq V(G\overline{G})$. Throughout this paper we consider the convex hull $H(S)$ most of times on the graph $G\overline{G}$. If we need to indicate the convex hull on another graph, say G, we add a subscript to the notation, e. g. $H_G(S)$.

Considering the geodetic hull number in complementary prisms, Coelho et al. [2016] presented bounds when G is a disconnected graph or a tree. Motivated by the works in [Coelho et al., 2016] and [Duarte et al., 2015] we present lower and upper bounds on the hull number of complementary prisms $G\overline{G}$ when both G and $\overline{G}$ are connected. In particular, we show that $h(G\overline{G}) \leq 3$ when G is a non-split graph. Otherwise, we characterize convex sets in split graphs G that allow the hull number for $G\overline{G}$ be unlimited.

This paper is divided in more three sections. In Section 2 we define the fundamental concepts. In Section 3 we present our main results. The conclusions follow in Section 4.

https://proceedings.science/p/85314?lang=pt-br

2. Preliminaries

Before we discuss our contributions, we present some relevant definitions. All graphs will be finite, simple, and undirected, and we use standard terminology and notation.

Let G be a graph. Given a vertex $v \in V(G)$, its open neighborhood is denoted by $N_G(v)$, and its *closed neighborhood*, denoted by $N_G[v]$, is the set $N_G[v] = N_G(v) \cup \{v\}$. For a set $U \subseteq V(G)$, let $N_G(U) = \bigcup_{v \in U} N_G(v) \setminus U$, and $N_G[U] = N_G(U) \cup U$. We denote the degree of a vertex $v \in V(G)$ by $deg_G(v)$. If $deg_G(v) = 0$ we say that v is an *isolated vertex*.

A K_n (resp. C_n) in a graph G denotes an induced complete graph (resp. cycle) on n vertices. A *clique* (resp. *independent set*) is a set of pairwise adjacent (resp. non-adjacent) vertices. A vertex of a graph G is *simplicial* in G if its neighborhood induces a clique.

The *distance* $d_G(u, v)$ of two vertices u and v in G is the minimum number of edges of a path in G between u and v. The greatest distance between any two vertices in G is the *diameter* of G, denoted by $diam(G)$. We say that H is an *isometric subgraph* of G if H is a subgraph of G such that $d_H(u, v) = d_G(u, v)$ for any pair $u, v \in V(H)$.

A graph G is called *connected* if any two of its vertices are linked by a path in G. Otherwise, G is called *disconnected*. A maximal connected subgraph of G is called a *connected component* or *component* of G. A component G_i of a graph G is *trivial*, if $|V(G_i)| = 1$, and *non-trivial* otherwise. If a graph G is connected and its complement $\overline{G}$ is also connected, we say that G is an *autoconnected* graph. The set of positive integers $\{1, \ldots, k\}$ is denoted by $[k]$.

A *split graph* G is one whose vertex set admits a partition $V(G) = C \cup I$ into a clique C and an independent set I. If such partition is unique, it is clear that C is a maximum clique and I is a maximum independent set. Otherwise, we consider the partition of $V(G)$ according to Observation 2.1.

Observation 2.1. *Let G be a split graph. If $V(G)$ admits a not unique partition into a clique C and an independent set I, we consider the partition $V(G) = C \cup I$ where*

(a) C is a maximum clique, if C is the unique maximum clique in G.

(b) I is a maximum independent set, otherwise.

3. Results

We begin by showing three fundamental lemmas and a proposition that we use in sequel.

Lemma 3.1 ([Dourado et al., 2009])**.** *Let G be a graph and S a proper and non-empty subset of $V(G)$. If $V(G) \setminus S$ is convex then every hull set of G contains at least one vertex of S.*

Lemma 3.2 ([Dourado et al., 2009])**.** *Let G be a graph and H an isometric subgraph of G. Then for every hull set S of H it holds that $V(H) \subseteq H_G(S)$.*

Lemma 3.3 ([Nascimento, 2016])**.** *Let G be a graph. If u is a simplicial vertex in G and $\overline{u}$ is a simplicial vertex in $\overline{G}$, then every hull set S of $G\overline{G}$ intersects $\{u, \overline{u}\}$.*

Proposition 3.4. *Let G be a graph, $S \subseteq V(G\overline{G})$, and $v_1 \ldots v_k$ be a path in G, for $k \geq 2$. If $\{v_1\} \cup \{\overline{v}_2, \ldots, \overline{v}_k\} \subseteq H(S)$, then $v_k \in H(S)$.*

Proof. The proof is by induction on k. First, let $k = 2$. Since $v_1v_2 \in E(G)$ and $v_1, \overline{v}_2 \in H(S)$, $v_2 \in I[v_1, \overline{v}_2]$. Now, let $k > 2$. Let $v_1 \ldots v_{k-1}v_k$ be a path in G and suppose that $\{v_1\} \cup \{\overline{v}_2, \ldots, \overline{v}_{k-1}, \overline{v}_k\} \subseteq H(S)$. By induction hypothesis $v_{k-1} \in H(S)$, which implies that $v_k \in I[v_{k-1}, \overline{v}_k]$. Therefore, it follows that $v_k \in H(S)$, for $k \geq 2$. □

Next we state our contributions. We first show that for complementary prisms of non-split graphs the geodetic hull number is limited by three.

https://proceedings.science/p/85314?lang=pt-br

Theorem 3.5. *Let G be a non-split autoconnected graph. Then $h(G\overline{G}) \leq 3$.*

Proof. Suppose that G is a non-split graph. According to Foldes and Hammer Foldes and Hammer [1977], G is split if and only if G does not have an induced subgraph isomorphic to one of the three forbidden graphs, C_4, C_5, or $2K_2$. To show the upper bound $h(G\overline{G}) \leq 3$ we construct hull sets on three cases: (1) C_4 or (2) C_5 or (3) $2K_2$ are induced subgraphs of G.

Case 1. Suppose that C_4 is an induced subgraph of G.

Let $V(C_4) = \{u_1, \ldots, u_4\}$, $E(C_4) = \{u_iu_{i+1} : 1 \leq i \leq 3\} \cup \{u_4u_1\}$, and $S = \{u_1, \overline{u}_3, \overline{u}_4\}$. We show that $H(S) = V(G\overline{G})$.

First, we show that $V(C_4) \cup \overline{V(C_4)} \subseteq H(S)$. We have that $u_4 \in I[u_1, \overline{u}_4]$ and $\overline{u}_1 \in I[u_1, \overline{u}_3]$. Consequently $u_3 \in I[u_4, \overline{u}_3]$ and $u_2 \in I[u_1, u_3]$, then $\overline{u}_2 \in I[u_2, \overline{u}_4]$. Hence $V(C_4) \cup \overline{V(C_4)} \subseteq H(S)$.

Let $A = (N_G(u_1) \cap N_G(u_3)) \cup (N_G(u_2) \cap N_G(u_4))$. Since $u_1u_3 \notin E(G)$ (resp. $u_2u_4 \notin E(G)$), then $v \in I[u_1, u_3]$ (resp. $v \in I[u_2, u_4]$), hence $A \subseteq H(S)$.

Let $B = V(G) \setminus (A \cup V(C_4))$. For every $b \in B$, there exist $i, j \in [4]$, $i \neq j$, such that $u_iu_j \in E(G)$ and $b \notin N_G(\{u_i, u_j\})$. Then $\overline{u}_i\overline{u}_j \notin E(\overline{G})$, consequently $\overline{b} \in I[\overline{u}_i\overline{u}_j]$, hence $\overline{B} \subseteq H(S)$.

Since G is connected, there exists a path in G joining every vertex from B to a vertex in $A \cup V(C_4)$. Since $\overline{B} \subseteq H(S)$, then Proposition 3.4 implies that $B \subseteq H(S)$. Hence $V(G) \subseteq H(S)$. Since $\overline{G}$ is connected, also by Proposition 3.4, $\overline{A} \subseteq H(S)$. Therefore S is a hull set of $G\overline{G}$.

Case 2. Suppose that C_5 is an induced subgraph of G.

Let $V(C_5) = \{u_1, \ldots, u_5\}$, $E(C_5) = \{u_iu_{i+1} : 1 \leq i \leq 4\} \cup \{u_5u_1\}$, and $S = \{u_1, u_4, \overline{u}_3\}$. We show that $H(S) = V(G\overline{G})$.

First, we show that $V(C_5) \cup \overline{V(C_5)} \subseteq H(S)$. We have that $u_5 \in I[u_1, u_4]$, $\overline{u}_1 \in I[u_1, \overline{u}_3]$ and $u_3 \in I[u_4, \overline{u}_3]$. Then $u_2 \in I[u_1, u_3]$. By Proposition 3.4 (dual), we have that $\overline{u}_5, \overline{u}_4, \overline{u}_2 \in H(S)$. Hence $V(C_5) \cup \overline{V(C_5)} \subseteq H(S)$.

Let $A = \{v \in N_G(\{u_i, u_j\}) : u_iu_j \notin E(G)\}$, for $i, j \in [5]$, $i \neq j$. By the definition of A, for every $v \in A$, there exists $i, j \in [5]$, $i \neq j$, such that $u_iu_j \notin E(G)$, which implies that $v \in I[u_i, u_j]$. Hence $A \subseteq H(S)$.

Now, let $B = V(G) \setminus (A \cup V(C_5))$. For every $b \in B$, there exist $i, j \in [5]$, $i \neq j$, such that $u_iu_j \in E(G)$ and $b \notin N_G(\{u_i, u_j\})$. Then $\overline{u}_i\overline{u}_j \notin E(\overline{G})$, thus $\overline{b} \in I[\overline{u}_i, \overline{u}_j]$. Hence $\overline{B} \subseteq H(S)$.

Since G is connected, there exists a path in G joining every vertex from B to a vertex in $A \cup V(C_5)$. Since $\overline{B} \subseteq H(S)$, then Proposition 3.4 implies that $B \subseteq H(S)$. Hence $V(G) \subseteq H(S)$. Since $\overline{G}$ is connected, also by Proposition 3.4, $\overline{A} \subseteq H(S)$. Therefore S is a hull set of $G\overline{G}$.

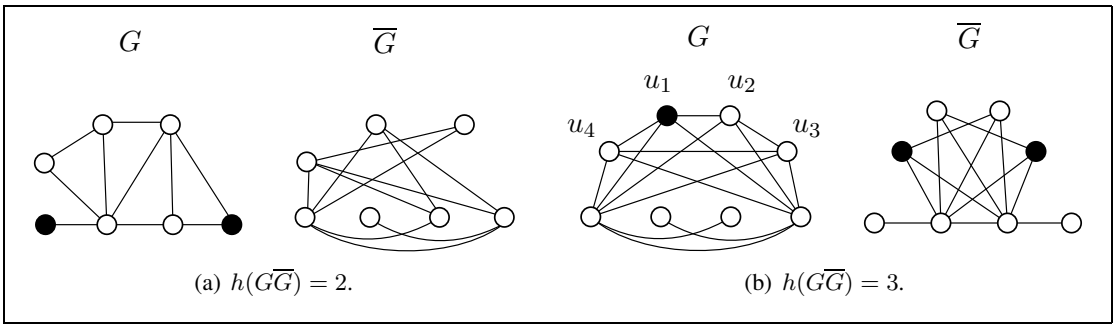
Case 3. Suppose that $2K_2$ is an induced subgraph of G.

Since $2K_2$ (resp. $G\overline{G}$) is isomorphic to $\overline{C}_4$ (resp. $\overline{G}G$), we consider $\overline{G}G$ and the proof follows by Case 1.

Since there exists a hull set of order three in all cases the upper bound $h(G\overline{G}) \leq 3$ holds for a non-split graph G. □

The bound of Theorem 3.5 is sharp, see Figure 1(b). The black vertices represent a hull set of each complementary prism $G\overline{G}$. For convenience, the edges joining corresponding vertices from G to $\overline{G}$ are not depicted in the figure.

https://proceedings.science/p/85314?lang=pt-br



(a) $h(G\overline{G}) = 2$. (b) $h(G\overline{G}) = 3$.

Figure 1: Graphs satisfying $h(G\overline{G}) \leq 3$.

3.1. Split Graphs

The following definitions and lemma characterize some convex sets in split graphs.

Let $G = (C \cup I, E)$ be a split graph. Let the graph G' arise from G by removing the edges of the graph induced by C. We call the graph G' as the *component graph* of G. We denote by $c(G')$ the number of connected components of G'. Furthermore, we refer to $nt(G')$ and $t(G')$ as the number of non-trivial and trivial components of G', respectively. Definition 3.6 expresses formally the idea of a component graph.

Definition 3.6. *Let $G = (C \cup I, E)$ be a split graph. We define the component graph G' of G as*

$$G' = G \setminus E(G[C]) = \bigcup_{i=1}^{c(G')} G_i.$$

Definition 3.7. *Let $G = (C \cup I, E)$ be a split autoconnected graph. We define the set $L(G_i)$ of a component G_i of G' as*

$$L(G_i) = V(G_i) \cup (V(\overline{G}_i) \cap \overline{C}),$$

for every $i \in [c(G')]$.

For an example of sets $L(G_i)$ of G', for $i \in [3]$, on the graph $G\overline{G}$, see Fig. 2.

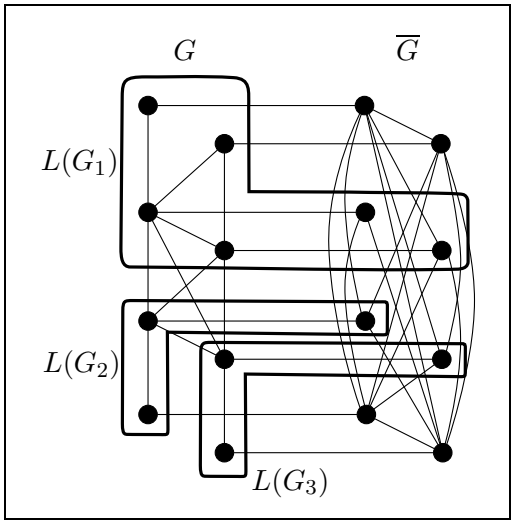


Figure 2: Sets $L(G_i)$ of G' on the graph $G\overline{G}$, for $i \in [3]$.

Lemma 3.8. *The set $V(G\overline{G}) \setminus L(G_i)$ is convex in $G\overline{G}$, for every $i \in [c(G')]$.*

https://proceedings.science/p/85314?lang=pt-br

Proof. Let $i \in [c(G')]$. We show that for all $w, w' \in V(G\overline{G}) \setminus L(G_i)$, $I[w, w'] \cap L(G_i) = \emptyset$. We have three cases:

1. $w, w' \in V(G) \setminus V(G_i)$;
2. $\overline{w}, \overline{w}' \in V(\overline{G}) \setminus (V(\overline{G}_i) \cap \overline{C})$;
3. $w \in V(G) \setminus V(G_i)$ and $\overline{w}' \in V(\overline{G}) \setminus (V(\overline{G}_i) \cap \overline{C})$.

Case 1. Let $w, w' \in V(G) \setminus V(G_i)$, and $v \in V(G_i)$.

Every (w, w')-path containing v has length at least 4, and thus it is not a shortest path since $diam(G\overline{G}) \leq 3$.

Case 2. Let $\overline{w}, \overline{w}' \in V(\overline{G}) \setminus (V(\overline{G}_i) \cap \overline{C})$, and $v \in L(G_i)$.

Let P be a shortest $(\overline{w}, \overline{w}')$-path passing through v. Clearly, $\bar{w}\bar{w}' \notin E(\overline{G})$ and therefore $ww' \in E(G)$.

If $v \in V(G_i)$, this implies that $v = w$ or $v = w'$, but this yields a contradiction since both w, w' cannot belong to I and one of w, w' must belong to $V(G_i) \cap C$. If $v \in V(\overline{G}_i) \cap \overline{C}$, $N_{\overline{G}}(v)$ is a clique, and thus P cannot be a shortest $(\overline{w}, \overline{w}')$-path, a contradiction.

Case 3. Let $w \in V(G) \setminus V(G_i)$, and $\overline{w}' \in V(\overline{G}) \setminus (V(\overline{G}_i) \cap \overline{C})$.

If $\overline{w}' = \overline{w}$, then $I[w, \overline{w}'] = \{w, \overline{w}'\}$. Now suppose that $\overline{w}' \neq \overline{w}$. If $ww' \notin E(G)$, then $I[w, \overline{w}'] = \{w, \overline{w}', \overline{w}\}$. Otherwise, $ww' \in E(G)$, then $I[w, \overline{w}'] = \{w, \overline{w}', w'\}$. Hence $I[w, \overline{w}'] \cap L(G_i) = \emptyset$.

By all cases, we obtain that for all $w, w' \in V(G\overline{G}) \setminus L(G_i)$, $I[w, w'] \cap L(G_i) = \emptyset$. Therefore, the set $V(G\overline{G}) \setminus L(G_i)$ is convex in $G\overline{G}$, for every $i \in [c(G')]$. □

Since $\overline{G}$ is also split, Lemma 3.8 holds for $\overline{G}G$, that is, considering $\bigcup_{j=1}^{c(\overline{G}')} G_j$ be the components of $\overline{G}'$, we have that the set $V(\overline{G}G) \setminus L(\overline{G}_j)$ is convex in $\overline{G}G$, for every $j \in [c(\overline{G}')]$. In view of that, we achieve the lower bound result.

Theorem 3.9. *Let G be a split autoconnected graph. Then $h(G\overline{G}) \geq \max\{c(G'), c(\overline{G}'), 2\}$.*

Proof. First, let G' or $\overline{G}'$ be disconnected graphs. By Lemma 3.8, $V(G\overline{G}) \setminus L(G_i)$, for every $i \in [c(G')]$, and $V(G\overline{G}) \setminus L(\overline{G}_j)$, for every $j \in [c(\overline{G}')]$, are convex sets. Thus, Lemma 3.1 implies that every hull set of $G\overline{G}$ must contain at least one vertex from $L(G_i)$ and $L(\overline{G}_j)$. Since the components G_i and $G_{i'}$, (resp. G_j and $G_{j'}$) are disjoints for all $i, i' \in [c(G')]$, $i \neq i'$, (resp. $j, j' \in [c(\overline{G}')]$, $j \neq j'$), each vertex $v \in V(G)$ (resp. $\overline{v} \in V(\overline{G})$) intersects exactly one $L(G_i)$ (resp. $L(G_j)$). This implies that $h(G\overline{G}) \geq \max\{c(G'), c(\overline{G}')\}$.

Now, let G' and $\overline{G}'$ be connected graphs. Then $c(G') = 1$ and $c(\overline{G}') = 1$, which implies that $\max\{c(G'), c(\overline{G}')\} = 1$. Since $|V(G)| \geq 2$, $h(G\overline{G}) \geq 2$, hence the result $h(G\overline{G}) \geq \max\{c(G'), c(\overline{G}'), 2\}$ holds. □

Proceeding on the upper bound results, first we present some useful lemmas. Let S be a subset of the vertices of a graph G. In Lemma 3.10 we show that for two vertices u, v in $V(G)$ at distance three from each other, if $u, v \in H(S)$, then their closed neighborhoods in G as well as their closed neighborhoods in $\overline{G}$ belong to $H(S)$. Further, we deal with a split graph G that has only one trivial component in G'. In Lemma 3.11 we show that if G' has only one trivial component, then $\overline{G}'$ also has only one trivial component, and vice-versa.

Lemma 3.10. *Let $G = (C \cup I, E)$ be a split autoconnected graph, $u, v \in I$ and $S \subseteq V(G\overline{G})$. If $d_G(u, v) = 3$ and $u, v \in H(S)$, then $N_G[\{u, v\}] \cup \overline{N_G[\{u, v\}]} \subseteq H(S)$.*

https://proceedings.science/p/85314?lang=pt-br

Proof. Let $u, v \in I$ such that $d_G(u, v) = 3$. Since $u\overline{uv}v$ and $uxyv$, for $x \in N_G(u)$ and $y \in N_G(v)$, are shortest paths between u and v, and $N_G(u) \cup N_G(v) \subseteq C$ we have that $N_G(u) \cup N_G(v) \cup \{\overline{u}, \overline{v}\} \subseteq I[u, v]$. Since $uy, vx \notin E(G)$, for $x \in N_G(u)$ and $y \in N_G(v)$, we have that $\bar{u}\bar{y}, \bar{v}\bar{x} \in E(\overline{G})$. Consequently $\overline{x} \in I[x, \overline{v}]$ and $\overline{y} \in I[y, \overline{u}]$. Therefore $N_G[\{u, v\}] \cup \overline{N_G[\{u, v\}]} \subseteq H(S)$. □

Lemma 3.11. *Let $G = (C \cup I, E)$ be a split autoconnected graph. Then $t(G') = 1$ if and only if $t(\overline{G}') = 1$.*

Proof. Suppose that $t(G') = 1$. Let v be the trivial component of G'. Since $N_G[v] = C$, $N_{\overline{G}}(\overline{v}) = \overline{I}$, hence $\overline{I} \cup \{\overline{v}\}$ is a clique of $\overline{G}$. Since $t(G') = 1$, for every $u \in C \setminus \{v\}$, $|N_G(u) \cap I| \geq 1$. This implies that, for every $\overline{u} \in \overline{C} \setminus \{\overline{v}\}$, $|N_{\overline{G}}(\overline{u})| \leq |\overline{I}| - 1$. Then $\overline{I} \cup \{\overline{v}\}$ is the unique maximum clique of $\overline{G}$.

Since C is maximum, for every $y \in I$, we have that $|N_G(y)| < |C|$ and since C is unique, there exists no $y \in I$ such that y is adjacent to $|C| - 1$ vertices. Then, for every $y \in I$, $|N_G(y)| \leq |C| - 2$. This implies that, for every $\overline{y} \in \overline{I}$, $|N_{\overline{G}}(\overline{y}) \cap \overline{C}| \geq 2$.

Since, for every $\overline{y} \in \overline{I}$, $N_{\overline{G}}(\overline{y}) \cap \overline{C} \neq \emptyset$, it follows that $\overline{v}$ is the unique trivial component of $\overline{G}'$, therefore $t(\overline{G}') = 1$.

Reciprocally, suppose that $t(\overline{G}') = 1$. Since $\overline{G}$ is also split, by the same previous arguments, now applied for $\overline{G}'$, we conclude that $t(G') = 1$. □

Let G be a split autoconnected graph. As we have done with the lower bounds, we establish a relation of the upper bounds of $h(G\overline{G})$ with the number of components of G'. To show that we divided the proof concerning about the number of trivial and non-trivial components of G'. The case of at least one of the parameters $nt(G'), t(G'), nt(\overline{G}')$ or $t(\overline{G}')$ is greater than one is provided by Theorem 3.12, and the remaining cases follow in Theorems 3.15 and 3.16.

Theorem 3.12. *Let $G = (C \cup I, E)$ be a split autoconnected graph. If $\max\{nt(G'), t(G'), nt(\overline{G}'), t(\overline{G}')\} \geq 2$, then $h(G\overline{G}) \leq \max\{c(G'), c(\overline{G}')\}$.*

Proof. To proceed with the proof we consider two cases (1) $nt(G') \geq 2$ or $nt(\overline{G}') \geq 2$, and (2) $t(G') \geq 2$ or $t(\overline{G}') \geq 2$. Since $\overline{G}$ is split, $G\overline{G}$ is isomorphic to $\overline{G}G$. Then, without loss of generality, we may consider $c(G') \geq c(\overline{G}')$. In view of that, we show a hull set of $G\overline{G}$ in each case: (1) $nt(G') \geq 2$ and (2) $t(G') \geq 2$.

Case 1. $nt(G') \geq 2$.

Let $\mathfrak{I}_1 = \{i \in [c(G')] : |V(G_i)| = 1\}$ and $\mathfrak{I}_2 = [c(G')] \setminus \mathfrak{I}_1$.

For every $k \in [c(G')]$, we obtain the set S by choosing $u_k \in V(G_k)$ such that $V(G_k) = \{u_k\}$ if $k \in \mathfrak{I}_1$ and $u_k \in V(G_k) \setminus C$ if $k \in \mathfrak{I}_2$. Notice that $|S| = c(G')$. We show that $H(S) = V(G\overline{G})$.

By hypothesis, $|\mathfrak{I}_2| \geq 2$. For all $i, j \in \mathfrak{I}_2$, since $d_G(u_i, v_j) = 3$, Lemma 3.10 implies that $N_G[\{u_i, v_j\}] \cup \overline{N_G[\{u_i, v_j\}]} \subseteq H(S)$

Let $u \in N_G(u_i)$, $v \in N_G(v_j)$, for some $i, j \in \mathfrak{I}_2$, $A = I \setminus N_G(u)$ and $B = I \setminus N_G(v)$. We have that $\overline{A} \subseteq N_{\overline{G}}(\overline{u})$, then $\overline{A} \subseteq I[\overline{u}, \overline{u}_i]$. Similarly, $\overline{B} \subseteq N_{\overline{G}}(\overline{v})$, then $\overline{B} \subseteq I[\overline{v}, \overline{v}_j]$. Since $A \cup B = I$, $\overline{I} \subseteq H(S)$.

Now, we show that for every $y \in I \setminus \{u_i, v_j\}$, $y \in H(S)$. Let $y \in I \setminus \{u_i, v_j\}$ such that $d_G(y, x) = 1$, for some $x \in C \cap H(S)$. Since G is connected x exists. Since $\overline{I} \cup \{x\} \subseteq H(S)$, Lemma 3.10 implies that $y \in H(S)$. Consequently $N_G(y) \subseteq I[y, x']$, for some $x' \in C$ such that $yx' \notin E(G)$. Hence $C \subseteq H(S)$. Again, by Lemma 3.10, $\overline{C} \subseteq H(S)$. Therefore S is a hull set of $G\overline{G}$, and $h(G\overline{G}) \leq c(G')$.

Case 2. $t(G') \geq 2$.

https://proceedings.science/p/85314?lang=pt-br

If $nt(G') \geq 2$ we fall back on Case 1. Thus, consider that $nt(G') = 1$. Let G_1 be the non-trivial component of G' and G_i, for $i \in [c(G')] \setminus \{1\}$, be the trivial components of G'. For some $x \in V(G_1) \cap C$, let $S = V(\overline{G_i}) \cup \{x\}$, for every $i \in [c(G')] \setminus \{1\}$. Notice that $|S| = c(G')$. We show that S is a hull set of $G\overline{G}$.

By hypothesis, $t(G') \geq 2$. Then there exist $i, j \in [c(G')] \setminus \{1\}$ such that G_i and G_j are trivial components in G'. Let v_i (resp. v_j) be the single vertex in $V(G_i)$ (resp. $V(G_j)$). Since $v_iv_j \in E(G)$, $\overline{v}_i\overline{v}_j \notin E(\overline{G})$. Consequently, $\overline{v}_i\overline{y}\overline{v}_j$ is a shortest $(\overline{v}_i, \overline{v}_j)$-path, for every $\overline{y} \in \overline{I}$. Then $\overline{I} \subseteq I[\overline{v}_i, \overline{v}_j]$.

Since $d_G(x, v_i) = 1$, for every $i \in [c(G')] \setminus \{1\}$, we have that $v_i \in I[x, \overline{v}_i]$. Since $d_G(x, y) = 1$, for every $y \in I \cap N_G(x)$, we obtain $y \in I[x, \overline{y}]$. Consequently $N_G(y) \subseteq I[y, v_i]$. Since $\overline{I} \subseteq H(S)$, by Lemma 3.10, $y' \in H(S)$. Consequently $C \subseteq H(S)$. Again, by Lemma 3.10, $\overline{C} \subseteq H(S)$. Therefore S is a hull set of $G\overline{G}$ and $h(G\overline{G}) \leq c(G')$.

By all cases there exists a hull set of $G\overline{G}$ of order $c(G')$. Since $c(G') \geq c(\overline{G}')$, the result $h(G\overline{G}) \leq \max\{c(G'), c(\overline{G}')\}$ holds. □

By Theorems 3.9 and 3.12 we obtain the following equality, stated in Corollary 3.13, which evidences that for complemetary prisms of split graphs the hull number can be unlimited.

Corollary 3.13. *Let $G = (C \cup I, E)$ be a split autoconnected graph. If $\max\{nt(G'), t(G'), nt(\overline{G}'), t(G')\} \geq 2$, then $h(G\overline{G}) = \max\{c(G'), c(\overline{G}')\}$.*

Proof. It follows directly from Theorems 3.9 and 3.12. □

Now, remains the cases in which $nt(G') < 2$, $nt(\overline{G}') < 2$, $t(G') < 2$ and $t(\overline{G}') < 2$. Since we disregard the trivial graphs, we have that $nt(G') = 1$ and $nt(\overline{G}') = 1$. By Lemma 3.11 if $t(G') = 1$, then $t(\overline{G}') = 1$. Otherwise, for $t(G') \neq 1$, we have that $t(\overline{G}') = 0$ or $t(\overline{G}') \geq 2$. If $t(\overline{G}') \geq 2$ the proof follows by Theorem 3.12 Case 2. So, it remains the cases $t(G') = 0$, $t(\overline{G}') = 0$, and $t(G') = 1$, $t(\overline{G}') = 1$, that follows in Theorems 3.15 and 3.16, respectively.

We first show a proposition that will be useful in the proof of Theorem 3.15. The next proposition is a slight modification of Proposition 3.4 now considering a split graph $G = (C \cup I, E)$. For a set of vertices S of $G\overline{G}$, Proposition 3.14 shows that every vertex lying on a path in the component graph G' of G will belong to $H(S)$ depending if some specific vertices in $C \cup \overline{I}$ belong to $H(S)$.

Proposition 3.14. *Let $G = (C \cup I, E)$ be a split graph. For $k \geq 2$, let $w_0v_1w_1 \ldots v_kw_k$ (or $w_0v_1w_1 \ldots v_k$) be a path in G', such that $v_i \in I$, for every $i \in [k]$, and $w_j \in C$, for every $j \in [k] \cup \{0\}$. Consider that, for every $i \in [k]$, there exists $z \in C$ such that $v_iz \notin E(G)$, and let $S \subseteq V(G\overline{G})$. If $\{w_0, z, \overline{v}_1, \ldots, \overline{v}_k\} \subseteq H(S)$, then $w_k \in H(S)$ (resp. $v_k \in H(S)$).*

Proof. The proof is by induction on k. Let $k = 1$. Since $w_0, \overline{v}_1 \in H(S)$, and $v_1\overline{v}_1 \in E(G\overline{G})$, we obtain that $v_1 \in I[w_0, \overline{v}_1]$. Since $z \in H(S)$, and $v_1z \notin E(G)$, then $w_1 \in I[v_1, z]$.

Let $k > 1$. By induction hypothesis, $w_{k-1} \in H(S)$. Then, since $\overline{v}_k \in H(S)$, we have that $v_k \in I[w_{k-1}, \overline{v}_k]$. Since $z \in H(S)$, and $v_kz \notin E(G)$, it follows that $w_k \in I[v_k, z]$. □

Theorem 3.15. *Let $G = (C \cup I, E)$ a split autoconnected graph such that $nt(G') = 1$, $nt(\overline{G}') = 1$, $t(G') = 0$ and $t(\overline{G}') = 0$. Then $h(G\overline{G}) \leq 3$.*

Proof. To prove that $h(G\overline{G}) \leq 3$ we must construct a hull set of $G\overline{G}$ of order three. We first show the existence of three vertices that are the basis of our construction.

Claim 1. There exist $y \in I$ such that $deg_G(y)$ is maximum, $x \in C \setminus N_G(y)$, and $y' \in N_G(x) \cap I$.

Proof of Claim 1. Since $nt(G') = 1$, $I \neq \emptyset$, then some vertex $y \in I$ has $deg_G(y)$ maximum.

https://proceedings.science/p/85314?lang=pt-br

Suppose, by contradiction, that there exists no $x \in C \setminus N_G(y)$. Then, for every $x \in C$, $x \in N_G(y)$. This implies that $C \cup \{y\}$ is a clique of G. If the partition of $V(G)$ is unique or selected according to Observation 2.1(a), C is the unique maximum clique in G, a contradiction. On the other hand, by Observation 2.1(b), I is a maximum independent set and there exist two or more maximum cliques in G. Let $v \in I$ such that $deg_G(v) = deg_G(y)$. Then, $C \cup \{v\}$ and $C \cup \{y\}$ are maximum cliques in G. Thus, we have that $\overline{I}$ is a maximum clique in $\overline{G}$. Let $\overline{w} \in \overline{C}$. Since $\bar{v}\bar{w}, \bar{w}\bar{y} \notin E(\overline{G})$, $(\overline{I} \setminus \{\overline{v}, \overline{y}\}) \cup \{\overline{w}\}$ is not a maximum clique in $\overline{G}$. Hence $\overline{I}$ is the unique maximum clique in $\overline{G}$. Selecting the partition of $V(\overline{G})$ according to Observation 2.1 (a), we obtain that $\overline{y}$ and $\overline{y}'$ are trivial components of $\overline{G}'$, a contradiction, since $t(\overline{G}') = 0$.

Since $t(G') = 0$ and $xy \notin E(G)$, x is not an isolated vertex in G', then there exists $y' \in N_G(x) \cap I$. □

Let $y \in I$ such that $deg_G(y)$ is maximum, $x \in C \setminus N_G(y)$, and $y' \in N_G(x) \cap I$. To avoid repetitive notation when considering the operations of union, intersection and symmetric difference on the sets $N_G(y)$ and $N_G(y')$, we write $U = N_G(y) \cup N_G(y')$, $A = N_G(y) \cap N_G(y')$, and $D = N_G(y) \triangle N_G(y')$, respectively.

We consider a partition $R \cup L$ of $I \setminus \{y, y'\}$ such that $R = \{v \in I \setminus \{y, y'\} : \forall u \in D,\ uv \in E(G)\}$, and $L = I \setminus (R \cup \{y, y'\})$. Furthermore, we let $R_1 = \{v \in R : N_G(v) \cap (C \setminus U) \neq \emptyset\}$, and $R_2 = R \setminus R_1$. We show hull sets of $G\overline{G}$ considering R_1 be empty or not.

Case 1. $R_1 = \emptyset$.

Let $S = \{x, y, \overline{y}'\}$. First, we have that $N_G(y) \subseteq I[x, y]$, $y' \in I[x, \overline{y}']$ and $\overline{y} \in I[y, \overline{y}']$.

Since $deg_G(y)$ is maximum, there exists $z \in N_G(y)$ such that $y'z \notin E(G)$. Thus $N_G(y') \subseteq I[y', z]$, which implies that $U \subseteq H(S)$. Since $D \subseteq U \subseteq H(S)$, and $\bar{w}\bar{y} \in E(\overline{G})$, for every $w \in N_G(y') \setminus A$, (resp. $\bar{w}'\bar{y}' \in E(\overline{G})$, for every $w' \in N_G(y) \setminus A$), we obtain that $\overline{D} \subseteq I[D \cup \{\overline{y}, \overline{y}'\}]$.

Let $v \in L$. By definition of L, there exists $z \in D$ such that $vz \notin E(G)$, which implies that $\bar{v}\bar{z} \in E(\overline{G})$. Since $\overline{z}$ is either adjacent to $\overline{y}$ or $\overline{y}'$, and, for every $\overline{v} \in \overline{L}$, $\bar{v}\bar{y}, \bar{v}\bar{y}' \in E(\overline{G})$, it follows that $\overline{v} \in I[\{\overline{y}, \overline{y}', \overline{z}\}]$. Hence $\overline{L} \subseteq H(S)$.

Now, we show that $L \cup (N_G(L) \setminus U) \subseteq H(S)$.

Since G' is connected, for $k \geq 2$, there exists a shortest path $w_0 v_1 w_1 \dots v_k w_k$ (or $w_0 v_1 w_1 \dots v_k$) in G' between $w_0 \in U$ and $w_k \in N_G(L) \setminus U$ (resp. $v_k \in L$), where $v_i \in I$, $w_i \in C$, for every $i \in [k]$.

By definition of R_2, no vertex in R_2 is adjacent to some vertex in $N_G(L) \setminus U$, then $w_0 v_1 w_1 \dots v_k w_k$ (or $w_0 v_1 w_1 \dots v_k$) does not contain vertices from R_2.

By the maximality of $deg_G(y)$ we have that $deg_G(v_i) < deg_G(y) + 1 = |N_G(y) \cup \{x\}|$, for every $i \in [k]$. Hence, there exists $z \in N_G(y) \cup \{x\}$ such that $v_i z \notin E(G)$, for every $i \in [k]$. Since $w_0, z \in U \subseteq H(S)$, and $\overline{v}_i \in \overline{I} \setminus \overline{R}_2 \subseteq H(S)$, for every $i \in [k]$, it follows from Proposition 3.14 that $w_k \in H(S)$ (resp. $v_k \in H(S)$). Consequently $L \cup (N_G(L) \setminus U) \subseteq H(S)$.

It remains to show that $R_2 \cup \overline{R}_2 \cup \overline{A} \cup \overline{N_G(L) \setminus U} \subseteq H(S)$.

Since $\overline{G}'$ is connected, for $k \geq 2$, there exists a shortest path $\overline{w}_0 \overline{v}_1 \overline{w}_1 \dots \overline{v}_k \overline{w}_k$ (or $\overline{w}_0 \overline{v}_1 \overline{w}_1 \dots \overline{v}_k$) in $\overline{G}'$ between $\overline{w}_0 \in \overline{I} \setminus \overline{R}_2$ and $\overline{w}_k \in \overline{R}_2$ (resp. $\overline{v}_k \in \overline{A} \cup \overline{N_G(L) \setminus U}$), where $\overline{v}_i \in \overline{C}$, and $\overline{w}_i \in \overline{I}$, for every $i \in [k]$.

We show that, for every $i \in [k]$, there exists $\overline{z} \in \overline{I} \setminus \overline{R}_2$ such that $\bar{v}_i \bar{z} \notin E(\overline{G})$. It is clear that $\overline{z} = \overline{y}$ for every $\overline{v} \in \overline{A}$, since $\bar{v}\bar{y} \notin E(\overline{G})$. Now, let $\overline{v} \in \overline{N_G(L) \setminus U}$. We know that $N_G(R_2 \cup \{y, y'\}) \cap (N_G(L) \setminus U) = \emptyset$. Since G' is connected, there exists $z \in L$ such that $vz \in E(G)$, which implies that $\bar{v}\bar{z} \notin E(\overline{G})$, with $\overline{z} \in \overline{L} \subseteq \overline{I} \setminus \overline{R}_2$.

Since $\overline{w}_0, \overline{z} \in \overline{I} \setminus \overline{R}_2 \subseteq H(S)$, and $v_i \in A \cup N_G(L) \subseteq H(S)$, for every $i \in [k]$, Proposition 3.14 (dual) implies that $\overline{w}_k \in H(S)$ (resp. $\overline{v}_k \in H(S)$). Hence, $\overline{R}_2 \cup \overline{A} \cup \overline{N_G(L) \setminus U} \subseteq H(S)$. Finally, since $D \cup \overline{R}_2 \subseteq H(S)$, we obtain that $R_2 \subseteq I[D \cup \overline{R}_2]$, which completes the proof of Case 1.

https://proceedings.science/p/85314?lang=pt-br

Case 2. $R_1 \neq \emptyset$.

Let $y'' \in R_1$ such that $N_G(y'') \setminus U$ is maximum, and $S = \{x, y, \overline{y}''\}$. First, we have that $N_G(y) \subseteq I[x, y]$, $y'' \in I[x, \overline{y}'']$, and $\overline{y} \in I[y, \overline{y}'']$. Since $deg_G(y)$ is maximum, y'' is not adjacent to every vertex in $N_G(y) \cup \{x\}$, then $N_G(y'') \subseteq I[N_G(y) \cup \{x, y''\}]$.

Let $x'' \in N_G(y'') \setminus U$. Since $x'' \in H(S)$, and $\bar{x}''\bar{y} \in E(\overline{G})$, we obtain that $\overline{x}'' \in I[x'', \overline{y}]$. Hence $\overline{N_G(y'') \setminus U} \subseteq H(S)$. Since $\bar{x}''\bar{y}' \in E(\overline{G})$, and $\bar{x}''\bar{y}'' \notin E(\overline{G})$, it follows that $\overline{y}' \in I[\overline{x}'', \overline{y}'']$. Consequently $y' \in I[x, \overline{y}']$. This implies that $\overline{D} \in I[D \cup \{\overline{y}, \overline{y}'\}]$. Since x'' is not adjacent to any vertex in R_2, we have that $\overline{x}''$ is adjacent to every vertex in $\overline{R}_2$. Thus, $\overline{R}_2 \subseteq I[\overline{x}'', \overline{y}'']$. Then, $R_2 \subseteq I[D \cup \overline{R}_2]$.

Let $v \in L$. By definition of L, there exists $z \in D$ such that $vz \notin E(G)$, which implies that $\bar{v}\bar{z} \in E(\overline{G})$. Since $\overline{z}$ is either adjacent to $\overline{y}$ or $\overline{y}'$, and, for every $\overline{v} \in \overline{L}$, $\overline{vy}, \bar{v}\bar{y}' \in E(\overline{G})$, it follows that $\overline{v} \in I[\{\overline{y}, \overline{y}', \overline{z}\}]$. Hence $\overline{L} \subseteq H(S)$.

We consider $X \cup Y$ a partition of $R_1 \setminus \{y''\}$ such that $X = \{v \in R_1 \setminus \{y''\} : \forall u \in (A \setminus N_G(y'')) \cup (N_G(y'') \setminus U),\ uv \in E(G)\}$, and $Y = R_1 \setminus (X \cup \{y''\})$.

Let $a \in A \setminus N_G(y'')$. Since $\bar{a}\bar{y}'' \in E(\overline{G})$, we obtain that $\overline{a} \in I[a, \overline{y}'']$. Consequently $\overline{A \setminus N_G(y'')} \subseteq H(S)$. Let $v \in Y$. By definition of Y, and since we selected $y'' \in R_1$ such that $N_G(y'') \setminus U$ is maximum, there exists $u \in (N_G(y'') \setminus U)$ such that $uv \notin E(G)$, then $\bar{u}\bar{v} \in E(\overline{G})$. Since $\bar{u}\bar{y}'' \notin E(\overline{G})$, and $\overline{u} \in \overline{N_G(y'') \setminus U} \subseteq H(S)$, we have that $\overline{v} \in I[\overline{u}, \overline{y}'']$. It follows that $\overline{Y} \subseteq H(S)$, and $Y \subseteq I[D \cup \overline{Y}]$. Hence, $N_G(Y) \subseteq H(S)$.

Now, we show that $L \cup (N_G(L) \setminus (N_G(Y \cup y'') \cup U)) \subseteq H(S)$.

Since G' is connected, for $k \geq 2$, there exists a shortest path $w_0 v_1 w_1 \dots v_k w_k$ (or $w_0 v_1 w_1 \dots v_k$) in G' between $w_0 \in N_G(Y \cup y'') \cup U$ and $w_k \in N_G(L) \setminus (N_G(Y \cup y'') \cup U)$ (resp. $v_k \in L$), where $v_i \in I$, $w_i \in C$, for every $i \in [k]$.

By definition of X, no vertex in X is adjacent to some vertex in $N_G(L) \setminus (N_G(Y \cup y'') \cup U)$, then $w_0 v_1 w_1 \dots v_k w_k$ (resp. $w_0 v_1 w_1 \dots v_k$) does not contain vertices from X.

By the maximality of $deg_G(y)$ we have that $deg_G(v_i) < deg_G(y) + 1 = |N_G(y) \cup \{x\}|$, for every $i \in [k]$. Hence, there exists $z \in N_G(y) \cup \{x\}$ such that $v_i z \notin E(G)$, for every $i \in [k]$. Since $w_0, z \in N_G(Y \cup y'') \cup U \subseteq H(S)$, and $\overline{v}_i \in \overline{I} \setminus \overline{X} \subseteq H(S)$, for every $i \in [k]$, it follows from Proposition 3.14 that $w_k \in H(S)$ (resp. $v_k \in H(S)$). Consequently $L \cup (N_G(L) \setminus (N_G(Y \cup y'') \cup U)) \subseteq H(S)$.

Similarly to Case 1, since $\overline{G}'$ is connected, Proposition 3.14 (dual) can be used to show that $\overline{X} \cup \overline{A \cap N_G(y'')} \cup \overline{N_G(L) \setminus (N_G(Y \cup y'') \cup U)} \subseteq H(S)$. Since $\overline{X} \subseteq H(S)$, we obtain that $X \subseteq I[\overline{X} \cup U]$, therefore $H(S) = V(G\overline{G})$.

By all cases we have that there exists a hull set of $G\overline{G}$ of order three, which completes the proof. □

Theorem 3.16. *Let $G = (C \cup I, E)$ a split autoconnected graph such that $nt(G') = 1$, $nt(\overline{G}') = 1$, $t(G') = 1$ and $t(\overline{G}') = 1$. Then $h(G\overline{G}) \leq 3$.*

Proof. Let v be the trivial component of G'. Let G_1 be the graph induced by $V(G) \setminus \{v\}$.

Since G_1 is a subgraph of G that satisfies $nt(G_1') = 1$, $nt(\overline{G}_1') = 1$, $t(G_1') = 0$ and $t(\overline{G}_1') = 0$, we proceed in a similar way as in the proof of Theorem 3.15. Let $y \in I$ such that $deg_G(y)$ is maximum, $x \in C \setminus (N_G(y) \cup \{v\})$, and $y' \in N_G(x) \cap I$. Claim 1 implies the existence of x, y, y' in $V(G_1) \subseteq V(G)$.

Let $U = N_G(y) \cup N_G(y')$, $A = N_G(y) \cap N_G(y')$, and $D = N_G(y) \triangle N_G(y')$. Consider $R = \{v \in I \setminus \{y, y'\} : \forall u \in D,\ uv \in E(G)\}$, and $R_1 = \{v \in R : N_G(v) \cap (C \setminus U) \neq \emptyset\}$. We show hull sets of $G\overline{G}$ considering R_1 be empty or not.

Case 1. $R_1 = \emptyset$.

https://proceedings.science/p/85314?lang=pt-br

Let $S = \{y, y', \overline{v}\}$. We have that $A = N_G(y) \cap N_G(y') \subseteq I[y, y']$. Since $nt(G') = 1$, we have that $d_G(y, y') < 3$, hence $A \neq \emptyset$. Since v is adjacent to every vertex in A, we obtain that $v \in I[A \cup \{\overline{v}\}]$. Consequently $U = N_G(y) \cup N_G(y') \subseteq I[\{v, y, y'\}]$. Since $vy, vy' \notin E(G)$, $\bar{v}\bar{y}, \bar{v}\bar{y}' \in E(\overline{G})$, then $\overline{y}, \overline{y}' \in I[\{y, y', \overline{v}\}]$.

At this point, we have that $x, y, \overline{y}' \in H(S)$. Since G_1 is a subgraph of G with $nt(G_1') = 1$, $nt(\overline{G}_1') = 1$, $t(G_1') = 0$ and $t(\overline{G}_1') = 0$, Theorem 3.15 Case 1 implies that $V(G_1) \cup V(\overline{G}_1) \subseteq H_{G_1\overline{G}_1}(\{x, y, \overline{y}'\})$. Since $G_1\overline{G}1$ is an isometric subgraph of $G\overline{G}$, Lemma 3.2 implies that $V(G_1) \cup V(\overline{G}_1) \subseteq H(\{x, y, \overline{y}'\})$, consequently $V(G\overline{G}) = H(S)$.

Case 2. $R_1 \neq \emptyset$.

Let $y'' \in R_1$ such that $N_G(y'') \setminus U$ is maximum, and $S = \{y, y'', \overline{v}\}$. We have that $N_G(y) \cap N_G(y'') \subseteq I[y, y']$. Since $nt(G') = 1$, we have that $d_G(y, y') < 3$, hence $N_G(y) \cap N_G(y'') \neq \emptyset$. Let $z \in N_G(y) \cap N_G(y'')$. Since $vz \in E(G)$, we have that $v \in I[z, \overline{v}]$. It follows that $N_G(y) \cup N_G(y'') \subseteq I[\{v, y, y''\}]$. Since $vy, vy'' \notin E(G)$, $\bar{v}\bar{y}, \bar{v}\bar{y}'' \in E(\overline{G})$, then $\overline{y}, \overline{y}'' \in I[\{y, y'', \overline{v}\}]$.

So far, we have that $x, y, \overline{y}'' \in H(S)$. Since G_1 is a subgraph of G with $nt(G_1') = 1$, $nt(\overline{G}_1') = 1$, $t(G_1') = 0$ and $t(\overline{G}_1') = 0$, Theorem 3.15 Case 2 implies that $V(G_1) \cup V(\overline{G}_1) \subseteq H_{G_1\overline{G}_1}(\{x, y, \overline{y}''\})$. Since $G_1\overline{G}1$ is an isometric subgraph of $G\overline{G}$, Lemma 3.2 implies that $V(G_1) \cup V(\overline{G}_1) \subseteq H(\{x, y, \overline{y}''\})$, consequently $V(G\overline{G}) = H(S)$.

By all cases we have that S is a hull set of $G\overline{G}$, and the result $h(G\overline{G}) \leq 3$ holds. □

We close our contributions by showing an equality for the case $nt(G') = 1$, $nt(\overline{G}') = 1$, $t(G') = 1$, expressed in Corollary 3.17.

Corollary 3.17. *Let $G = (C \cup I, E)$ be a split autoconnected graph. If $nt(G') = 1$, $nt(\overline{G}') = 1$, $t(G') = 1$ and $t(\overline{G}') = 1$, then $h(G\overline{G}) = 3$.*

Proof. Let v be the trivial component of G'. The upper bound follows by Theorem 3.16. So, remains to show the lower bound $h(G\overline{G}) \geq 3$.

Since v is a simplicial vertex in G, and $\overline{v}$ is a simplicial vertex in $\overline{G}$, Lemma 3.3 implies that $S \cap \{v, \overline{v}\} \neq \emptyset$. Let $x \in \{v, \overline{v}\}$, and suppose that $S = \{u, x\}$ is a hull set of $G\overline{G}$. Since $d_{G\overline{G}}(u, x) < 2$, for every $u \in V(G\overline{G}) \setminus \{v, \overline{v}\}$, we have that $H(S) = N_{G\overline{G}}[u] \cap N_{G\overline{G}}[x]$. This implies that $H(S) \neq V(G\overline{G})$, a contradiction. Therefore, $h(G\overline{G}) \geq 3$. □

4. Conclusion

We have considered the geodetic hull number on complementary prisms $G\overline{G}$ of both connected graphs G and $\overline{G}$. When G is a split graph, we have shown lower and upper bounds showing that the hull number is unlimited. For non-split graphs, we have shown that the hull number is at most three. As future work, we suggest the study of other convexity parameters of geodetic convexity for complementary prisms.

Acknowledgements

The authors would like to thank CAPES and FAPEG for the partial support.

References

Albenque, M. and Knauer, K. (2014). Convexity in partial cubes: the hull number. In *LATIN 2014: Theoretical Informatics*, p. 421–432. Springer.

Araujo, J., Campos, V., Giroire, F., Nisse, N., Sampaio, L., and Soares, R. (2013). On the hull number of some graph classes. *Theoretical Computer Science*, 475:1–12.

Centeno, C. C., Dourado, M. C., Penso, L. D., Rautenbach, D., and Szwarcfiter, J. L. (2011). Irreversible conversion of graphs. *Theoretical Computer Science*, 412(29):3693–3700.

https://proceedings.science/p/85314?lang=pt-br

Coelho, E. M. M., Coelho, H., Nascimento, J. R., and Szwarcfiter, J. L. (2016). O número envoltório de prismas complementares na convexidade geodética. In *Anais do XLVIII Simpósio Brasileiro de Pesquisa Operacional*, volume 1, p. 1–10.

Domingos, P. and Richardson, M. (2001). Mining the network value of customers. In *Proceedings of the Seventh ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '01, p. 57–66, New York, NY, USA. ACM. ISBN 1-58113-391-X.

Dourado, M. C., Gimbel, J. G., Kratochvíl, J., Protti, F., and Szwarcfiter, J. L. (2009). On the computation of the hull number of a graph. *Discrete Mathematics*, 309(18):5668–5674.

Dourado, M. C., Penso, L. D., and Rautenbach, D. (2016). On the geodetic hull number of P_k-free graphs. *Theoretical Computer Science*, 640:52–60. ISSN 0304-3975.

Dourado, M. C., Protti, F., Rautenbach, D., and Szwarcfiter, J. L. (2010a). On the hull number of triangle-free graphs. *SIAM Journal on Discrete Mathematics*, 23(4):2163–2172.

Dourado, M. C., Protti, F., and Szwarcfiter, J. L. (2010b). Complexity results related to monophonic convexity. *Discrete Applied Mathematics*, 158(12):1268–1274.

Dourado, M. C. and Sampaio, R. M. (2016). Complexity aspects of the triangle path convexity. *Discrete Applied Mathematics*, 206:39–47.

Dreyer, P. A. and Roberts, F. S. (2009). Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion. *Discrete Applied Mathematics*, 157(7):1615–1627.

Duarte, M. A., Penso, L., Rautenbach, D., and dos Santos Souza, U. (2015). Complexity properties of complementary prisms. *Journal of Combinatorial Optimization*, p. 1–8. ISSN 1573-2886.

Everett, M. G. and Seidman, S. B. (1985). The hull number of a graph. *Discrete Mathematics*, 57 (3):217–223.

Foldes, S. and Hammer, P. L. (1977). Split graphs. In *Proceedings 8th Southeastern Conference on Combinatorics, Graph Theory and Computing, Louisiana State University, Baton Rouge, LA*, p. 311–315.

Hammack, R., Imrich, W., and Klavžar, S. (2011). *Handbook of Product Graphs*. CRC press.

Haynes, T. W., Henning, M. A., Slater, P. J., and van der Merwe, L. C. (2007). The complementary product of two graphs. *Bulletin of the Institute of Combinatorics and its Applications*, 51:21–30.

Hernando, C., Jiang, T., Mora, M., Pelayo, I. M., and Seara, C. (2005). On the Steiner, geodetic and hull numbers of graphs. *Discrete Mathematics*, 293(1):139–154.

Kanté, M. M. and Nourine, L. (2016). Polynomial time algorithms for computing a minimum hull set in distance-hereditary and chordal graphs. *SIAM Journal on Discrete Mathematics*, 30(1): 311–326.

Nascimento, J. R. (2016). O Número Envoltório P_3 e o Número Envoltório Geodético em Produtos de Grafos. Master's thesis, Universidade Federal de Goiás.

Peleg, D. (2002). Local majorities, coalitions and monopolies in graphs: a review. *Theoretical Computer Science*, 282(2):231–257.

https://proceedings.science/p/85314?lang=pt-br